EXPONENTIAL RADON TRANSFORM INVERSION BASED ON HARMONIC ANALYSIS OF THE EUCLIDEAN MOTION GROUP

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ABSTRACT

This paper presents a new method for the exponential Radon transform inversion based on harmonic analysis of the Euclidean motion group ($M(2)$). The exponential Radon transform is modified to be formulated as a convolution over $M(2)$. The convolution representation leads to a block diagonalization of the modified exponential Radon transform in the Euclidean motion group Fourier domain, which provides a deconvolution type inversion for the exponential Radon transform. Numerical examples are presented to show the viability of the proposed method.

1. INTRODUCTION

For a uniform attenuation coefficient $\mu \in \mathbb{C}$, the exponential Radon transform of a compactly supported real valued function f over $\mathbb{R}^2$ is defined as

$$\mathcal{T}_\mu f(\boldsymbol{\theta}, t) = \int_{\mathbb{R}^2} f(\boldsymbol{x})\delta(\boldsymbol{x}\cdot\boldsymbol{\theta} - t)\mathrm{e}^{\mu \boldsymbol{x}\cdot\boldsymbol{\theta}^\perp} d\boldsymbol{x}, \tag{1}$$

where $t \in \mathbb{R}$, $\boldsymbol{\theta} = (\cos\theta, \sin\theta)^T$ is a unit vector on S^1 with $\theta \in [0, 2\pi)$ and $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta)^T$.

The exponential Radon transform constitutes a mathematical model for imaging modalities such as x-ray tomography ($\mu = 0$), single photon emission tomography (SPECT) ($\mu \in \mathbb{R}$) [7], and optical polarization tomography of stress tensor field ($\mu \in \mathrm{i}\mathbb{R}$) [8].

A number of different approaches have been proposed for the exponential Radon transform inversion. These can be classified into three categories: Fourier, filtered back projection and circular harmonic decomposition [1, 9, 3, 2, 4, 6] type inversion methods. A unified framework and a detailed review of the inversion methods can be found in [6].

This paper presents an alternative approach and provides a new inversion method for the exponential Radon transform based on the harmonic analysis of the Euclidean motion group, denoted by $M(2)$. This method of inversion leads to new algorithms for the inversion of the exponential Radon transform. The reconstructed images for μ real and μ imaginary are presented to show the viability of the proposed method.

2. MODIFIED EXPONENTIAL RADON TRANSFORM AS A CONVOLUTION OVER THE GROUP $M(2)$

The rigid motions of $\mathbb{R}^2$ form a group called the Euclidean motion group denoted by $M(2)$. The elements of the group are the 3×3 dimensional matrices of the form

$$(R_\theta, \boldsymbol{r}) = \begin{bmatrix} R_\theta & \boldsymbol{r} \\ \boldsymbol{0}^T & 1 \end{bmatrix}, \tag{2}$$

where $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2)$ is the rotation component and $\boldsymbol{r} = (r_1, r_2)^T \in \mathbb{R}^2$ is the translation component. The group operation is the usual matrix multiplications and inverse of an element is obtained by matrix inversion as $(R_\theta, \boldsymbol{r})^{-1} = (R_\theta^{-1}, -R_\theta^{-1}\boldsymbol{r})$.

Let $g = (R_\theta, \boldsymbol{r})$ and $h = (R_\phi, \boldsymbol{x})$. Convolution over $M(N)$ is defined as

$$(f_1 *_{M(2)} f_2)(g) = \int_{M(2)} f_1(h) f_2(h^{-1}g) d(h) \tag{3}$$

where $\int_{M(2)} d(h) = \int_{SO(2)} \int_{\mathbb{R}^2} d\boldsymbol{x} d(\phi)$. Then, multiplying the exponential Radon transform of a real valued function f with $\mathrm{e}^{\mu r_2}$, $r_2 \in \mathbb{R}$, the resulting integral can be expressed as a convolution operation over $M(2)$ as follows:

$$\begin{aligned} \mathcal{T}'_\mu f(g) &= \mathrm{e}^{\mu r_2}\mathcal{T}_\mu f(\boldsymbol{\vartheta}, -r_1) \\ &= \int_{\mathbb{R}^2} f(\boldsymbol{x})\delta(\boldsymbol{x}\cdot\boldsymbol{\vartheta} + r_1)\mathrm{e}^{\mu \boldsymbol{x}\cdot\boldsymbol{\vartheta}^\perp + \mu r_2} d\boldsymbol{x} \\ &= (\Lambda_\mu *_{M(2)} f^*)(g), \end{aligned} \tag{4}$$

where $f^*(h) = \overline{f(h^{-1})}$ and $\Lambda_\mu(h) = \delta(\boldsymbol{x}\cdot\boldsymbol{e_1})\mathrm{e}^{\mu \boldsymbol{x}\cdot\boldsymbol{e_2}}$. Λ_μ will be called the convolution filter.

We shall use the Fourier transform over the group $M(2)$ to express the modified exponential Radon transform, $\mathcal{T}'_\mu f$, as a multiplication in the $M(2)$-Fourier domain.

3. $M(2)$-FOURIER TRANSFORM

Let $f \in L^2(M(2))$. The $M(2)$-*Fourier transform* of f is defined as

$$\mathcal{F}_{M(2)}(f)_{mn}(\lambda) = \widehat{f}_{mn}(\lambda) = \int_{M(2)} f(g) u_{mn}^{(\lambda)}(g^{-1}) d(g), \tag{5}$$

for $\lambda \geq 0$, and the corresponding *inverse* $M(2)$-*Fourier transform* is given by

$$\mathcal{F}_{M(2)}^{-1}(\widehat{f}_{mn})(g) = f(g) = \int_0^\infty \sum_{m,n} \widehat{f}_{mn}(\lambda) u_{nm}^{(\lambda)}(g) \lambda d\lambda, \tag{6}$$

0-7803-9134-9/05/$20.00

where $u^{(\lambda)}_{mn}(g)$ are the matrix elements of the irreducible unitary representations of $M(2)$ given by [10]

$$u^{(\lambda)}_{mn}(g) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-im\omega}\mathrm{e}^{-\mathrm{i}\lambda(r_1\cos\omega + r_2\sin\omega)}\mathrm{e}^{in(\omega-\theta)}d\omega. \tag{7}$$

Let $f,\ f_1,\ f_2 \in L^2(M(2))$. Then, $M(2)$-Fourier transform transforms satisfies the following properties:

1. Adjoint property:
$$\widehat{f^*}_{mn}(\lambda) = \overline{\widehat{f}_{nm}(\lambda)}, \tag{8}$$
where $f^*(g) = \overline{f(g^{-1})}$.
2. Convolution property:
$$\mathcal{F}(f_1 * f_2)_{mn}(\lambda) = \sum_q \widehat{f}_{2mq}(\lambda)\,\widehat{f}_{1qn}(\lambda). \tag{9}$$
3. If f is an $SO(2)$ invariant function over $M(2)$, i.e. $f(g) = f(\boldsymbol{r}) \in L^2(\mathbb{R}^2)$, then
$$\widehat{f}_{mn}(\lambda) = \delta_m \widetilde{f}_n(-\lambda) \tag{10}$$
where δ_m is the Kronecker delta function.

4. INVERSION OF EXPONENTIAL RADON TRANSFORM USING $M(2)$-FOURIER TRANSFORM

Treating the projections $\mathcal{T}'f_\mu$ and the filter Λ_μ as distributions over $M(2)$, the modified exponential Radon transform can be expressed as a multiplication in the $M(2)$-Fourier transform domain. Using the convolution property of the $M(2)$-Fourier transform, (4) becomes:

$$\widehat{\mathcal{T}'_\mu f}_{mn}(\lambda) = \sum_q \overline{\widehat{f}_{qm}(\lambda)}\widehat{\Lambda_\mu}_{qn}(\lambda) = \overline{\widehat{f}_{0m}(\lambda)}\widehat{\Lambda_\mu}_{0n}(\lambda), \tag{11}$$

for $m, n \in \mathbb{Z}$. Equation 11 provides a block diagonal representation of the modified exponential Radon transform in the $M(2)$-Fourier domain, where each block is parameterized by $\lambda \geq 0$. It also provides an immediate inversion formula for the exponential Radon transform:

$$f = \mathcal{F}^{-1}_{M(2)}\left(\sum_k \left[\widehat{\mathcal{T}'_\mu f}_{nk}\left[\widehat{\Lambda_\mu}\right]^{-1}_{km}\right]^\dagger\right). \tag{12}$$

Since $\widehat{\Lambda_\mu}$ is rank one, as long as $\widehat{\Lambda_\mu}_{0n}(\lambda) \neq 0$, by (11), the $M(2)$-Fourier coefficients of f is given by

$$\widehat{f}_{0m}(\lambda) = \overline{\left(\frac{\widehat{\mathcal{T}'_z f}_{mn}(\lambda)}{\widehat{\Lambda_\mu}_{0n}(\lambda)}\right)}. \tag{13}$$

Thus, the inversion formula for the exponential Radon transform becomes:

$$\begin{aligned} f(\boldsymbol{x}) &= \mathcal{F}^{-1}_{M(2)}\left(\delta_k \widetilde{f}_{-m}(-\lambda)\right) = \mathcal{F}^{-1}\left(\delta_k \overline{\left(\frac{\widehat{\mathcal{T}'_\mu f}_{mn}(\lambda)}{\widehat{\Lambda_\mu}_{0n}(\lambda)}\right)}\right) \\ &= \int_0^\infty \sum_m \overline{\left(\frac{\widehat{\mathcal{T}'_\mu f}_{mn}(\lambda)}{\widehat{\Lambda_\mu}_{0n}(\lambda)}\right)} u^{(\lambda)}_{m0}(h)\lambda^{N-1}d\lambda. \end{aligned} \tag{14}$$

5. RECONSTRUCTION ALGORITHMS

Two algorithms are proposed for exponential Radon transform inversion. The algorithms differ mainly in the way the $M(2)$-Fourier coefficients, $\widehat{f}_{0m}(\lambda)$, of the function are computed (step 3 of the reconstruction.). In the first algorithm, (14) is used to compute $\widehat{f}_{0m}(\lambda)$, whereas in the second algorithm, (12) is used. Let $f(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > a$ and hence $\mathcal{T}_\mu f(\theta, r_1) = 0$ for $|r_1| > a$. Proposed exponential Radon transform inversion can be implemented in four steps:

Step 1. Extend $\mathcal{T}_\mu f(\theta, -r_1)$ to $\mathcal{T}'_\mu f(g)$ by multiplying with $\mathrm{e}^{\mu r_2}$ for $r_2 \in [-a, a]$, where $0 < a < \infty$.

Step 2. Compute $\widehat{\mathcal{T}'_\mu f}_{mn}(\lambda)$, the $M(2)$-Fourier transform of $\mathcal{T}'_\mu f$ for $m, n = 0, \pm1, \ldots, \pm S - 1$, and $\lambda_k = \frac{k\lambda_0}{S}$, $k = 0, \ldots, S-1$ for some $\lambda_0 > 0$.

Step 3. Compute $\widetilde{f}_{-m}(-\lambda)$ by either of the following ways:

Algorithm 1. For each λ, let $[\widehat{\Lambda}_0(\lambda)]$ and $[\widehat{\mathcal{T}'_\mu f}_m(\lambda)]$ denote the row vectors with their elements given by $\widehat{\Lambda}_{0n}(\lambda)$ and $\widehat{\mathcal{T}'_\mu f}_{mn}(\lambda)$, respectively. For each m, compute $\widetilde{f}_{-m}(-\lambda)$ by

$$\widetilde{f}_{-m}(-\lambda) = \frac{\overline{[\widehat{\mathcal{T}'_\mu f}_m(\lambda)][\widehat{\Lambda}_0(\lambda)]^T}}{\overline{[\widehat{\Lambda}_0(\lambda)]}[\widehat{\Lambda}_0(\lambda)]^T + \sigma}, \tag{15}$$

where σ is a positive constant close to zero.

Algorithm 2. For each λ, let $[\widehat{\Lambda}(\lambda)]$, $[\widehat{\mathcal{T}'_\mu f}(\lambda)]$ and $[\widehat{f}(\lambda)]$ denote the matrices with their corresponding elements given by $\delta_m\widehat{\Lambda}_{0n}(\lambda)$, $\widehat{\mathcal{T}'_\mu f}_{mn}(\lambda)$ and $\delta_m\widetilde{f}_{-n}(-\lambda)$, where m and n denote the row and column numbers, respectively. Then,

$$[\widehat{f}(\lambda)] = [\widehat{\Lambda}(\lambda)]\left(\overline{[\widehat{\Lambda}(\lambda)]}^T[\widehat{\Lambda}(\lambda)] + \sigma I\right)^{-1}\overline{[\widehat{\mathcal{T}'_\mu f}(\lambda)]}^T, \tag{16}$$

where σ is a positive constant close to zero. Note that a generalization of *Algorithm 2* for $\mu = 0$ was presented in our earlier work [11].

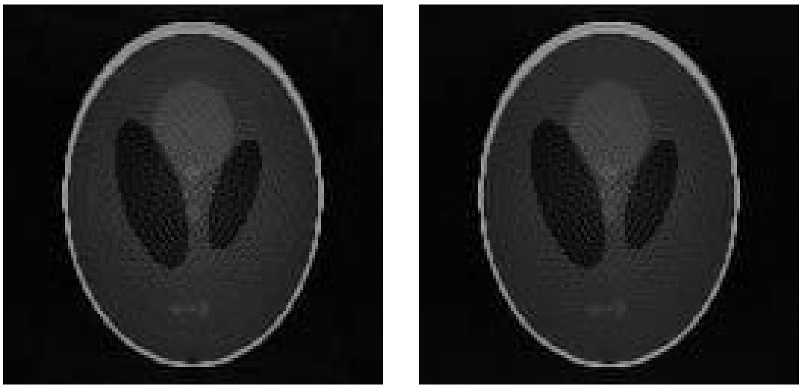
Step 4. Using Equation 10, form $\widehat{f}_{mn}(\lambda)$ and take the inverse $M(2)$-Fourier transform to obtain f.

In step 3, $\widehat{\Lambda_\mu}_{0n}$ can be very close to zero, making the inverse filtering numerically unstable. Therefore, the inverse filter is replaced with its regularized linear least square version to stabilize the inversion.

A fast implementation of the $M(2)$-Fourier transform is implemented as described in [5, 11].

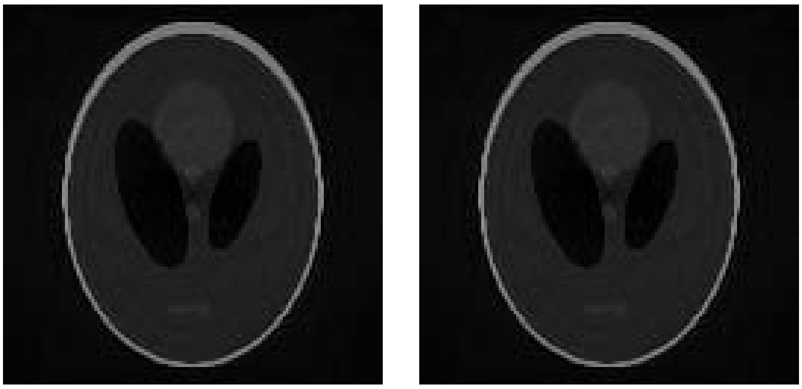
6. NUMERICAL SIMULATIONS

Numerical simulations are performed on a two-dimensional modified Shepp-Logan phantom image of $13.1 \times 13.1\text{cm}^2$, discretized by 129×129 pixels. The $M(2)$-Fourier transform was numerically implemented as described above. All numerical implementations were performed using MATLAB. Figure 1 and 2 present the reconstructed images using the proposed algorithms for $\mu = 0.154\text{cm}^{-1}$ and $\mu = \mathrm{i}0.154\text{cm}^{-1}$, respectively. In all reconstructions, σ is set to 10^{-10}. The case for $\mu = 0$ was studied previously in [11, 12]. These simulations demonstrate the viability of the proposed method and algorithms.



(a) (b)

Fig. 1. Reconstruction of the modified Shepp-Logan phantom using the proposed algorithms for $\mu = 0.154\text{cm}^{-1}$. (a) *Algorithm 1*, $\sigma = 10^{-10}$. (b) *Algorithm 2*, $\sigma = 10^{-10}$.



(a) (b)

Fig. 2. Reconstruction of the modified Shepp-Logan phantom using the proposed algorithms for $\mu = \text{i}0.154\text{cm}^{-1}$. (a) *Algorithm 1*, $\sigma = 10^{-10}$. (b) *Algorithm 2*, $\sigma = 10^{-10}$.

7. CONCLUSION

In this paper, a block diagonal form of the modified exponential Radon transform in the $M(2)$-Fourier domain is derived and a new method of inversion for the exponential Radon transform using the $M(2)$-Fourier transform is introduced. Numerically stable algorithms are introduced. Viability of proposed method and algorithms is demonstrated by numerical simulations.

8. REFERENCES

[1] Bellini S, Piancentini M, Cafforio C, Rocca F 1979 *IEEE Transactions on Acoustics, Speech, and Signal Processing* **ASSP-27** 213–218

[2] Hawkins G W, Leichner P K, Yang, N-C 1988 *IEEE Transactions on Medical Imaging* **7** 135–148

[3] Inouye T, Kose K, Hasegawa A 1989 *Phys. Med. Biol.* **34** 299–304

[4] Kuchment P, Shneiberg I 1994 *Applicable Analysis* **53** 221–231

[5] Kyatkin A B, Chirikjian G S 2000 *Applied Computational Harmonic Analysis* **9** 220–241

[6] Metz C E, Pan X 1995 *IEEE Transactions on Medical Imagin* **14** 643–658

[7] Natterer F 1986 *The Mathematics of Computerized Tomography* (New York: Wiley-Teubner)

[8] Puro, A 2001 *Inverse Problems* **17** 179–1888

[9] Tretiak O, Metz C 1980 *SIAM J. Appl. Math.* **39** 341–354

[10] Vilenkin N J 1988 *Special functions and the theory of representations* (Providence: American Mathematical Society)

[11] C.E. Yarman, B. Yazıcı, "Radon transform inversion via Wiener filtering over the Euclidean motion group," *Proceedings of IEEE international Conference on Image Processing, vol 2*, Barcelona, Spain, pp. 811–814, 2003.

[12] C.E. Yarman, B. Yazıcı, 2004 "Radon transform inversion based on harmonic analysis of the motion group," (preprint)