

Hopf Bifurcation Control of Power Systems Nonlinear Dynamics Via a Dynamic State Feedback Controller—Part I: Theory and Modelling

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Abstract—Power systems exhibit voltage collapse at saddle-node bifurcation (SNB), which is a static bifurcation. However, it has been proved that before reaching SNB, which also represents the maximum loadability of the system, voltage collapse can occur due to oscillatory behavior of the system caused by Hopf bifurcations in stationary branch of the solutions. This two-part paper introduces a dynamic state feedback control law which guarantees the elimination of Hopf bifurcations before occurrence of the SNB. Part I is devoted to the mathematical representation of the detailed system dynamics, investigation of Hopf bifurcation and SNB theorems, and state feedback controller design. For the purpose of dynamical analysis, stable equilibrium of the system is obtained. Then the control system is designed with the objective of preventing the voltage collapse before the SNB, such that the structural stability of the system is preserved in the stationary branch of the solutions. The controller aims to relocate Hopf bifurcations to stationary branch of solutions located after SNB, so that the possibility of voltage collapse is eliminated from normal operating region of the system. To obtain a detailed nonlinear model of power systems, saturation phenomenon has been integrated in the nonlinear dynamics of a three node power system. In order to evaluate the performance of the proposed controller, bifurcation analysis has been performed in Part II using single-machine and multi-machine test systems.

Index Terms—Bifurcation theory, Hopf bifurcation, nonlinear dynamics, synchronous generator saturation, state feedback control, voltage collapse.

I. INTRODUCTION

Power systems are one of the most complex physical systems. The complexity is not only originated from large number of equations describing the system behavior, but also in the nonlinearity of equations. Nonlinear nature of the system is because of the highly nonlinear dynamics of synchronous generators, and also the dynamic loads of the system. Similar to all the dynamical systems, power systems are described

P. Mahdavi-pour Vahdati is supported by the EUs H2020 under grant agreement no 674875. Innovative Training Network oCPS.

L. Vanfretti was supported in part by the StandUP for Energy Collaboration Initiative and the ITEA3 project 14018 - OPENCPs.

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by sets of differential equations, complexity of which is contingent upon modeling precision of the system.

One of the main objectives of dynamical studies in power systems is the prevention of catastrophic failures. Such failures might lead to destructive blackouts in the power systems [1]. Furthermore, the security issues regarding future power systems have been comprehensively studied in [2].

Saturation is a nonlinear phenomenon and it could lead to unexpected variations in operating modes of the system. Incorporation of the saturation provides a more accurate dynamical model in both sub transient and transient operating modes of the power systems. However, this incorporation is a computationally expensive task, since saturation is a local phenomenon in synchronous generators. To this end, analytical modeling of the saturation is required. In the previous studies many modeling methods have been introduced [3], [4]. The saturation model introduced in [5] is employed for the dynamical studies of this paper. Since saturation is a highly nonlinear phenomenon, robustness against parameter variations and accuracy of the saturation model are key factors, especially for the purpose of bifurcation analysis that the system experiences various modes of operation and bifurcations.

Several approaches towards the investigation of voltage collapse have been introduced in the literature [6]–[8]. A theoretical approach towards investigation of voltage collapse mechanism is presented in [6]. It has been shown, by using center manifold theorem, that occurrence of the saddle-node bifurcation (SNB) leads to voltage collapse in the system. However, the SNB is a static bifurcation, and dynamic bifurcations also can lead to voltage collapse as discussed in [8]. According to [8], it has been shown that dynamic bifurcations, most importantly Hopf Bifurcation, can lead to collapse of the system before the occurrence of SNB. Furthermore, chaotic behavior caused by the periodic orbits emerged from Hopf bifurcations is observed in [7], [9]. In addition to [7]–[9], in [10] a numerical study has been performed on two test cases. It has been concluded that in more realistic power system models, the observation of dynamic bifurcations leading to chaotic dynamics is more probable.

Control of dynamic bifurcations in the power systems equilibria contributes to the prevention of voltage collapse before reaching the SNB. Many studies have been carried out to eliminate Hopf bifurcations. In [11], FACTS devices have been effectively utilized for elimination of dynamic bifurcations and chaos in power systems. Effectiveness of FACTS devices

on control and elimination of Hopf bifurcations have been further investigated in [12], [13]. Application of classical control methods in systems with severe nonlinearities is also a challenging task. In [14], local bifurcation control problems are defined and employed in the study of the local feedback stabilization problem for nonlinear systems in critical cases. In [15], a model-based control strategy based on the global state feedback linearization has been employed for elimination of chaotic behavior from a model power system. However, the saturation phenomenon is not considered and the proposed control law is static. According to [16], dynamic state feedback control law with incorporation of washout filters effectively contributes in elimination of the undesired Hopf bifurcations. In [17], a static state feedback control law with polynomial functions is introduced. However, the proposed control laws are able to control only one Hopf bifurcation.

In order to obtain a comprehensive dynamical model of the power systems, the saturation phenomenon is considered. To the best of authors' knowledge, integration of *saturation* in the power systems dynamics for the purpose of bifurcation analysis and control, has not been investigated in the previous studies. Moreover, previous studies focused on controlling only *one* Hopf bifurcation. Note that the system nonlinearities is considerably increased by taking saturation into account.

In this two-part paper, a dynamic state feedback control law is proposed which guarantees the elimination of Hopf bifurcations before occurrence of SNB. The proposed dynamic state feedback controller is motivated by a controller proposed by Wang and Abed in [16] for bifurcation control of a chaotic system. Furthermore, the nonlinear nature of saturation is included in a power system model consisting of a synchronous generator, dynamic load, and an infinite bus. The general framework of this paper is illustrated in Fig. 1. As depicted in Fig.1, first, the dynamical equations of components in the power system under analysis are derived and assembled. The saturation model is included in the assembly phase. The system is represented by a system of Differential Algebraic Equations (DAEs), and algorithms for bifurcation analysis of DAE systems are used. However, it is possible to derive the Ordinary Differential Equations (ODEs) representing the system of DAEs and to use ODE methods for bifurcation analysis. The approach used in Part I is to assemble a model of ODEs, while in Part II the generic application to a multi-machine system is done by assembling a DAE model. Because the methodology does not depend on the model representation, the choice of an ODE or DAE representation depends mainly in the modeling effort involved. In the case of an ODE representation, as done in Part I, this is more challenging as power system analysis tools mostly use DAE representations, so explicit ODE models have to be developed for scratch. Next, the equilibria structure of the system is derived by finding the stationary solutions of the system (either ODE or DAE), and a stable equilibrium is chosen for initiation of the bifurcation analysis. From the results of the bifurcation analysis, Hopf bifurcations (HBs) and Saddle-node bifurcation (SNB) are detected. In case there are two Hopf bifurcations, the control design would include two control gain vectors for effective relocation of Hopf bifurcations. Otherwise one control gain

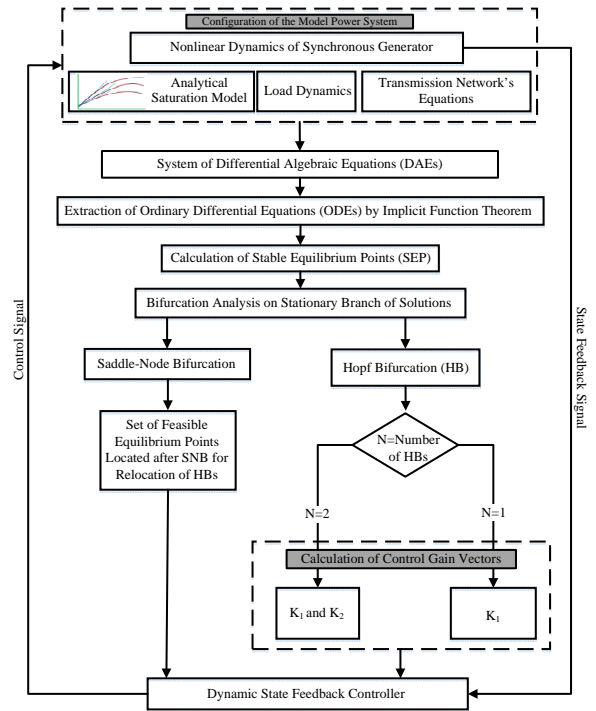


Fig. 1. General framework of the proposed nonlinear state feedback controller for Hopf bifurcation control

vector would be sufficient. In this study, Hopf bifurcations are relocated from stationary branch of solutions before SNB, to stationary branch located after SNB. Hence, the dynamic state-feedback controller would require the equilibria branch of the system. After the control design, the corresponding state for which the state-feedback control has been designed is updated and system stability is ensured until reaching SNB, which is the physical stability limit of the system. The main contribution of this paper is three-fold:

- A dynamic state feedback control law has been introduced which guarantees the elimination of Hopf bifurcations before occurrence of saddle-node bifurcation (SNB). In other words, voltage collapse is prevented before reaching the SNB to preserve the structural stability of the system.
- The control law relocates Hopf bifurcations to stationary branch of solutions located after the SNB.
- The saturation phenomenon is integrated into the nonlinear dynamics of the power systems. This integration provides a more detailed dynamical behavior representation of the power systems, i.e. bifurcation characteristic of the system is affected by the saturation phenomenon.

The rest of this paper is organized as follows. In Section II, dynamic modeling of power systems is investigated. Furthermore, the inclusion of saturation phenomenon is introduced and a high order dynamical model is obtained. In Section III, differential algebraic equations are discussed and the system dynamics are reduced to obtain ordinary differential equations. Section IV provides a detailed introduction of bifurcation theory with corresponding theorems. In Section V, the dynamic

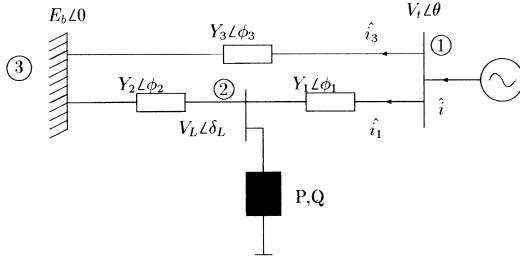


Fig. 2. The single-machine system model used for this study [20]

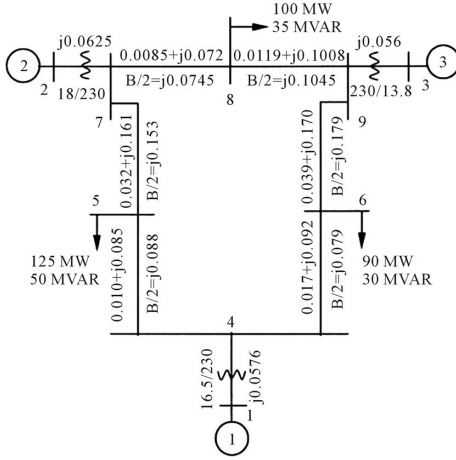


Fig. 3. The multi-machine system model used for this study [18]

state feedback controller which is used for elimination of Hopf bifurcations from stationary branch of solutions is introduced. Section VI concludes the part I of the paper. Mathematical proofs of the theorems are provided in the Appendices.

II. POWER SYSTEMS NONLINEAR DYNAMICS

In this section, nonlinear dynamics of power systems is obtained. In fact, there are different approaches towards modeling the dynamics of power systems. In single-machine dynamics for incorporation of saturation phenomenon in nonlinear dynamics of power systems, the *electric current* model of the synchronous generator dynamics is employed. In this model the system of differential equations for currents of synchronous generator are used. Integration of saturation is performed, and nonlinear dynamics of synchronous generator are obtained. The structures of the systems under study are presented in Figs. 2 and 3. The dynamical modeling of the multi-machine systems is provided in [18]. Also, the data used for modeling of different elements of the system are provided in part II [19].

A. Synchronous Generator Dynamics

Traditional approach for magnetic saturation modeling of salient-pole synchronous machines is based on the assumption that the degree of d-axis and q-axis saturations are the same. Since q-axis is not accessible in practice, the d-axis magnetizing curves were used and in fact accuracy of modeling proved to be poor; particularly for the case of salient-pole machines [5]. In the following method, which is based on steady-state

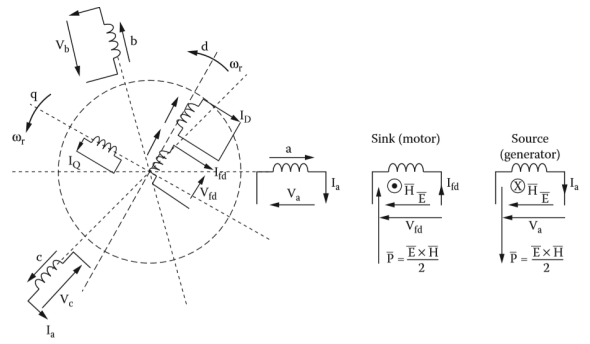


Fig. 4. Synchronous generator and Poynting vectors [21]

operating data, it is proved by simulations that the Reciprocity Condition has no significance in modeling of synchronous generator in transient and steady-state modes of operation [5]. Here, analytical model of saturation is presented.

1) *Analytical Saturation Model*: The salient-pole synchronous generator is modeled in d-axis and q-axis via Park's Transformation. A typical generator consisting of two windings in d-axis and one winding in q-axis, is depicted in Fig. 4 including the corresponding Poynting vectors [21]. The direction of the currents in Fig. 4, are derived from the direction of the magnetic and electric fields in Poynting vectors.

First, the saturation characteristics relating the magnetizing currents in direct and quadrature axes of generator to synchronous inductances should be calculated. The d-axis and q-axis magnetizing currents in per-unit are as

$$\begin{cases} i_{md} = -i_d + i_{mf} + i_D \\ i_{mq} = -i_q + i_{mf} \end{cases} \quad (1)$$

where i_D and i_Q are the damper winding currents in corresponding axes. The saturation characteristics which are modeled by the polynomial functions of two variables, are obtained as [5]

$$\begin{cases} L_d = \sum_{j=1}^m (j+1) i_{md}^{j-1} \sum_{k=1}^m c_{jk} i_{mq}^{k+1} + C_1 \\ L_q = \sum_{j=1}^m i_{md}^{j+1} \sum_{k=1}^m c_{jk} (k+1) i_{mq}^{k-1} + C_2 \end{cases} \quad (2)$$

where m is a positive integer representing the order of polynomial functions. In addition, c_{jk} , C_1 , and C_2 are the polynomial coefficients.

By neglecting the reciprocity condition, (2) could be simplified without reducing the accuracy of the model.

$$\begin{cases} L_d = \sum_{j=0}^m i_{md}^j \sum_{k=0}^n a_{jk} i_{mq}^k \\ L_q = \sum_{j=0}^m i_{mq}^j \sum_{k=0}^n b_{jk} i_{md}^k \end{cases} \quad (3)$$

Also, a_{jk} and b_{jk} , are the polynomial coefficients. The generator parameters can be found in [5].

2) *Incorporation of Saturation in Synchronous Generator Dynamics*: The direction of the stator currents are considered

to be out of the terminals. Accordingly, matrix form of voltage equations is obtained as [5]

$$\frac{d\psi}{dt} = \mathbf{v} - \mathbf{R}\mathbf{i} + \mathbf{e}, \quad (4)$$

where:

$$\begin{aligned} \psi &= [\psi_d, \psi_{mf}, \psi_{D1}, \psi_q, \psi_{Q1}]^T, \\ \mathbf{v} &= [v_d, v_f, 0, v_q, 0]^T, \\ \mathbf{i} &= [-i_d, i_{mf}, i_{D1}, -i_q, i_{Q1}]^T, \\ \mathbf{e} &= [\omega_r \psi_q, 0, 0, -\omega_r \psi_d, 0]^T, \\ \mathbf{R} &= \text{diag}(R_s, R_f, R_D, R_s, R_Q). \end{aligned} \quad (5)$$

Also, motion equation is expressed as

$$\frac{d\omega_r}{dt} = \frac{1}{2H}(T_L + \psi_q i_d - \psi_d i_q), \quad (6)$$

where H denotes the inertia constant and T_L is the externally applied mechanical torque.

In order to avoid an algebraic loop in dynamical analysis, currents are chosen as state variables. In II-A1, analytical modeling of saturation was introduced. Flux linkages are expressed as $\psi = \mathbf{L}\mathbf{i}$. Also, it was shown that inductances are functions of magnetizing currents. Consequently, time derivation of ψ , which is needed for restatement of equations in terms of currents, could be expressed as [5]

$$\frac{d\psi}{dt} = (\mathbf{L} + \mathbf{L}_D) \frac{d\mathbf{i}}{dt}, \quad (7)$$

where \mathbf{L} represents the matrix of static inductances and \mathbf{L}_D is the matrix of dynamic inductances. Mutual inductances of d-axis and q-axis are calculated as

$$\begin{cases} L_{md} = L_d - L_{ls} \\ L_{mq} = L_q - L_{ls} \end{cases} \quad (8)$$

where L_d and L_q are calculated in (3). Hence, the dynamic inductances matrix is derived as [5]

$$\mathbf{L}_D = \begin{bmatrix} \frac{\partial L_{md}}{\partial i_{md}} \mathbf{G}_{33} & \frac{\partial L_{md}}{\partial i_{mq}} \mathbf{G}_{32} \\ \frac{\partial L_{mq}}{\partial i_{md}} \mathbf{G}_{23} & \frac{\partial L_{mq}}{\partial i_{mq}} \mathbf{G}_{22} \end{bmatrix}. \quad (9)$$

In (9), \mathbf{G}_{33} , \mathbf{G}_{32} , \mathbf{G}_{23} and \mathbf{G}_{22} , are unit matrices. According to previous derivations, state equations with currents as states can be expressed in the following form [5]:

$$\frac{d\mathbf{i}}{dt} = (\mathbf{L} + \mathbf{L}_D)^{-1}(\mathbf{v} - \mathbf{R}\mathbf{i} + \mathbf{e}). \quad (10)$$

Equation (10), represents synchronous generator dynamics with inclusion of saturation dynamics. It also contains 5 states of dynamics. Rotor angle of the generator and equation of motion give another two states of the system as

$$\begin{aligned} \frac{d\delta}{dt} &= \omega_b(\omega_r - 1), \\ \frac{d\omega_r}{dt} &= \frac{1}{2H}(T_L + \psi_q i_d - \psi_d i_q). \end{aligned} \quad (11)$$

It should be mentioned that all equations are in per-unit. Detailed dynamics of a small power system including one dynamic load consists of 9 differential equations, 7 of which are dynamics of synchronous generator. The system of equations forms a system of differential algebraic equations. The other two equations, which describe load dynamics, are introduced in the following.

B. Nonlinear Load Dynamics

Dynamic load consists of an induction motor, in parallel with a static load. The differential equations describing its behavior are as [6], [20]

$$\begin{cases} P = P_{ld} + P_0 + p_1 \dot{\theta}_L + p_2 \dot{V}_L + p_3 V_L \\ Q = Q_{ld} + Q_0 + q_1 \dot{\theta}_L + q_2 \dot{V}_L + q_3 V_L^2 \end{cases} \quad (12)$$

In this load model, angle and amplitude of load bus voltage are considered as states.

C. Algebraic Equations of System

Here, algebraic equations of the system are presented.

$$\begin{cases} \hat{v}_t = (v_d + jv_q)e^{j(\delta - \frac{\pi}{2})} \\ \hat{i} = (i_d + ji_q)e^{j(\delta - \frac{\pi}{2})} \\ \theta = \delta - \frac{\pi}{2} + \tan^{-1}\left(\frac{v_q}{v_d}\right) \\ P = v_l i_d \sin(-\theta_l + \delta) + v_l i_q \cos(-\theta_l + \delta) \\ \quad + v_l y_2 e_m \cos(-\theta_l + \Phi_2) - v_l^2 y_2 \cos(\Phi_2) \\ Q = v_l i_d \cos(-\theta_l + \delta) - v_l i_q \sin(-\theta_l + \delta) \\ \quad - v_l y_2 e_m \sin(-\theta_l + \Phi_2) + v_l^2 y_2 \sin(\Phi_2) \\ P_G = v_d i_d + v_q i_q \\ Q_G = v_q i_d - v_d i_q \end{cases} \quad (13)$$

The algebraic equations provide the basic formulations to eliminate the algebraic variables of the system, as nonlinear functions of the states of the system.

III. DIFFERENTIAL ALGEBRAIC MODEL OF SYSTEM

The differential and algebraic equations expressed in previous sections form a system of differential algebraic equations (DAE). In this study, DAEs are assumed to be smooth. The system is expressed in general terms as

$$\begin{aligned} \dot{x} &= f(x, y, \mu), \\ 0 &= g(x, y, \mu), \end{aligned} \quad (14)$$

where x is the vector of state variables, y denotes the vector of algebraic variables, and μ is the free parameters of the system. The system is defined by (14), is considered *theoretically problematic*, since the system of algebraic equations may contain singular points. According to [22], at any singular point, the constraint manifold of the system, which is a vector field satisfying the algebraic section of the equations, cannot be well defined. For the purpose of nonlinear analysis, it is convenient to reduce the differential algebraic model of the system to autonomous ordinary differential equations. This is done by employing Implicit Function Theorem, which is explained below. Henceforth, D denotes the derivative operator.

Theorem 1 (Implicit Function Theorem). *Let F be in $C^1(\Omega; \mathbb{R}^m)$ where $\Omega \in \mathbb{R}^n \times \mathbb{R}^m$ is an open set. Also, let (a, b) denote a point in Ω such that $\frac{\partial F}{\partial y}(a, b)$ is invertible, and $F(a, b) = 0$. Then, there exists an open set $\mathcal{H} \in \mathbb{R}^n$ which contains a , and an open set $\mathcal{Y} \in \mathbb{R}^m$ satisfying the following.*

- For all $x \in \mathcal{H}$, there exist a unique $y = f(x) \in \mathcal{Y}$ such that $F(x, f(x)) = 0$.
- $f(a) = b$, Also, $f: \mathcal{H} \rightarrow \mathcal{Y}$ is of class C^1 and we have:

$$Df(x) = - \left[\frac{\partial F}{\partial y}(x, f(x)) \right]_{m \times m}^{-1} \left[\frac{\partial F}{\partial x}(x, f(x)) \right]_{m \times n}, \quad \forall x \in \mathcal{H}. \quad (15)$$

If the Jacobian of algebraic equations is non-singular, then by Implicit Function Theorem $g(x, y)$ can be inverted locally and substituted $\dot{x} = f(x, y)$ in (14). For the system introduced in (14), consider a point (x, y, μ) for which the algebraic Jacobian $g_x(x, y, \mu)$ is nonsingular. According to *Theorem 1*, a locally unique smooth function F exists with the following form with no algebraic variables:

$$\dot{x} = F(x, y, \mu). \quad (16)$$

In dynamics introduced in (10) and (11) substituting v_d and v_q from (17), eliminates the algebraic variables. Hence, the system can be expressed by a set of ordinary differential equations (ODEs).

$$\begin{aligned} v_d &= (\sin(-\Phi_1 - \theta_l + \delta) \cos(\Phi_1) y_1 v_l \\ &\quad + \sin(\Phi_1) \cos(-\Phi_1 - \theta_l + \delta) y_1 v_l \\ &\quad + \sin(\Phi_1) i_q + \cos(\Phi_1) i_d)(y_1)^{-1}, \\ v_q &= -(\sin(-\Phi_1 - \theta_l + \delta) y_1 v_l \sin(\Phi_1) \\ &\quad - \cos(\Phi_1) \cos(-\Phi_1 - \theta_l + \delta) y_1 v_l \\ &\quad + i_d \sin(\Phi_1) - \cos(\Phi_1) i_q)(y_1)^{-1}. \end{aligned} \quad (17)$$

Equation (17), is the solution to the algebraic part of the system, which is replaced in ODEs of the system to eliminate algebraic variables of the system. In other words, using the equations above, the algebraic variables of the system are eliminated and the system of DAEs is transformed to a system of ODEs describing the dynamics of the system.

A. Reduced Jacobian Matrix and Equilibria of the System

For a fixed value of μ , the stationary solutions of (14) which also are the equilibria of the system, are expressed as

$$\begin{aligned} f(x, y, \mu) &= 0, \\ g(x, y, \mu) &= 0. \end{aligned} \quad (18)$$

Let J denote the Jacobian matrix of the system (14), which is in the following form:

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}. \quad (19)$$

By linearization of (14) around the equilibrium, the stability of the equilibria can be determined from eigenvalues of the Jacobian matrix:

$$\begin{bmatrix} \Delta \dot{x} \\ 0 \end{bmatrix} = J \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}. \quad (20)$$

Based on *Theorem 1*, if g_y is non-singular, Δy can be eliminated by substitution of algebraic variables in differential equations. Hence, the linearized system can be represented as

$$\Delta \dot{x} = [f_x - f_y g_y^{-1} g_x] \Delta x. \quad (21)$$

From (21),

$$A = F_x = [f_x - f_y g_y^{-1} g_x]. \quad (22)$$

A is the Jacobian matrix of (16), and the Schur complement¹ of the g_y in the unreduced Jacobian matrix [22]. Therefore, A

¹For matrix A represented in the block form of $\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right]$, Schur complement corresponding to A_{22} is determined by: $A/A_{22} = A_{11} - A_{12}A_{22}^{-1}A_{21}$.

is the reduced Jacobian matrix of the system. Stability of the equilibria in stationary branch of solutions, derived from (18), are determined by the eigenvalues of the reduced Jacobian matrix of the system. Eigenvalues of (16) are obtained from roots of the polynomial shown in (23).

$$A - \lambda I = 0, \quad (23)$$

where A denotes the reduced Jacobian matrix evaluated at a specific equilibrium obtained from (18). Stable equilibrium point (SEP) of a dynamical system, is defined as an equilibrium at which all the eigenvalues of the Jacobian matrix obtained from (23) have negative real parts.

IV. BIFURCATION THEORY

Bifurcation is defined as the qualitative change in behavior of a deterministic dynamical system, caused by quasi-static variations in one or more parameters of the system. There are several types of bifurcations. In this section, local bifurcations in stationary branch of solutions are introduced. Consider the following dynamical system

$$\dot{x} = F(x, \mu). \quad (24)$$

By variation of μ , at a specific point namely μ_0 , the system experiences bifurcation and loses its structural stability [23]. μ_c represents the bifurcation value. Saddle-node bifurcation theorem is given below.

Theorem 2. (Saddle-Node Bifurcation Theorem) Assume that (24) has an equilibrium at $\mu = \mu_0$. The following conditions hold for this equilibrium:

- $D_x F(p, \mu_0)$ denotes the Jacobian of F with respect to x with a single eigenvalue 0, right eigenvector v , and left eigenvector w . Moreover, $D_x F(p, \mu_0)$ has k eigenvalues with negative real parts and $n - k - 1$ eigenvalues with positive real parts,
- $\langle w, D_\mu F(p, \mu_0) \rangle \neq 0$,
- $\langle w, D_{xx} F(p, \mu_0)(v, v) \rangle \neq 0$.

Then there is a smooth curve of equilibria in $\mathbb{R}^n \times \mathbb{R}$, passing through (p, μ_0) and tangent to $\mathbb{R}^n \times \{\mu_0\}$ at (p, μ_0) . Hence, the SNB occurs at (p, μ_0) .

Here, the Hopf bifurcation theorem is stated.

Theorem 3. (Hopf Bifurcation Theorem) Consider (24) with the following properties:

- For $\mu \rightarrow 0$, there exists an equilibrium at the origin with

$$D_x F(0, \mu) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \quad (25)$$

for smooth functions $\alpha = \alpha(\mu)$, $\beta = \beta(\mu)$ with $\alpha(0) = 0$ and $\beta(0) \neq 0$. Note that $D_x F(0, 0)$ has complex conjugate eigenvalues shown as $\pm \beta i$;

- $\alpha'(0) > 0$;
- For

$$F(x, y, \mu) = \begin{bmatrix} \alpha x - \beta y + g(x, y, \mu) \\ \beta x + \alpha y + h(x, y, \mu) \end{bmatrix} \quad (26)$$

the first Lyapunov coefficient calculated as

$$\begin{aligned} a &= \frac{-1}{16\beta} [(g_{xxx} + g_{xyy} + h_{xxy} + h_{yyx}) \beta + g_{xy}(g_{xx} \\ &\quad + g_{yy}) - h_{xy}(h_{xx} + h_{yy}) - g_{xx}h_{xx} + g_{yy}h_{yy}]. \end{aligned} \quad (27)$$

Note that $a \neq 0$ at $(x, y, \mu) = (0, 0, 0)$.

The Hopf bifurcation occurs with the following properties:

- 1) If $a > 0$, there is a supercritical Hopf bifurcation. For $\mu \leq 0$ the origin is stable and attracts all nearby orbits, while for $\mu > 0$ the origin is unstable and there is a stable periodic orbit that attracts all nearby orbits except for the origin.
- 2) If $a < 0$, there is a subcritical Hopf bifurcation. For $\mu < 0$ there is an unstable periodic orbit; the stable equilibrium at the origin attracts only points inside the orbit, while for $\mu \geq 0$ the origin is unstable and no orbits stay close to the origin.

Given that the primary source of chaos could be attributed to emergence of periodic orbits by Hopf bifurcations in power systems, elimination of Hopf bifurcations preserves the structural stability, and prevents the system collapse before violating the physical limits [23], [24].

A. System Dynamics

From II-B and II-A2, the state variables of the single-machine system could be introduced as

$$x = [i_d, i_{mf}, i_{D1}, i_q, i_{Q1}, \omega, \delta, v_l, \theta_l]^T. \quad (28)$$

Also, free parameters used for bifurcation analysis for the single-machine system are:

$$\mu = [P_{ld}, Q_{ld}]^T. \quad (29)$$

As for the case of multi-machine system, the state variables are as [18]:

$$x = [\delta_1, \omega_1, E'_{q1}, \delta_2, \omega_2, \delta_3, \omega_3]^T. \quad (30)$$

Also, free the parameter used for bifurcation analysis in this case is:

$$\mu = \lambda. \quad (31)$$

where λ represents the load of the system. It should be noted that for the purpose of control performance evaluation, the above-mentioned dimensions of dynamics for the multi-machine system is sufficient.

Hence, the system is defined by following form of DAE:

$$\dot{x} = F(x, \mu). \quad (32)$$

V. DYNAMIC STATE-FEED BACK CONTROLLER FOR HOPF BIFURCATION CONTROL

In this section, a dynamic state feedback controller is proposed for anti-control of Hopf bifurcations. The objective of anti-control of Hopf bifurcation is to design a controller by which the qualitative behavior of the system could be modified, and a desired bifurcation characteristic for the system could be achieved. One of the most commonly used anti-control methods, is to create bifurcations at specific and desired locations on stationary branch of solutions. This leads to changes in qualitative behavior of the system. By creating bifurcations at appropriate locations with suitable control methods, the qualitative behavior of the system could be modified [17], [25]. One of the widely accepted methods for

anti-control of Hopf bifurcations is the state feedback control method. The proposed dynamic state feedback controller, does not alter the equilibria structure of the system which is an essential characteristic for bifurcation control objectives [26].

Analytical determination of conditions which lead to inception of a Hopf bifurcation, needs expressions of eigenvalues of the Jacobian matrix to be derived. However, for high order systems derivation of analytical expressions for eigenvalues is a computationally infeasible task, or in some cases even impossible. In order to avoid such analytical derivations, the equivalent criteria for occurrence of Hopf bifurcation which is introduced in [27], is employed.

A. Detection of Hopf Bifurcations

Let $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ denote the nonlinear smooth function representing the dynamics of n -dimensional system, $\dot{x} = F(x, \mu)$, where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. Assume that (x^e, μ^e) is an equilibrium of the system, i.e. $F(x^e, \mu^e) = 0$. Also, let $J(x^e, \mu^e) = \frac{\partial F}{\partial x}(x = x^e, \mu = \mu^e)$ show the Jacobian matrix of the system evaluated at this equilibrium. Equivalent criteria for emergence of Hopf bifurcation is derived by the coefficients of characteristic polynomial of Jacobian matrix as

$$P(\lambda; \mu^e) = \det(\lambda I_n - J(x^e, \mu^e)) = p_0(\mu^e)\lambda^n + p_1(\mu^e)\lambda^{n-1} + \dots + p_n(\mu^e), \quad (33)$$

where I_n is n -dimensional identity matrix. The following matrix is obtained by modifying (33).

$$H_n(\mu^e) = \begin{bmatrix} p_1(\mu^e) & p_0(\mu^e) & \cdots & 0 \\ p_3(\mu^e) & p_2(\mu^e) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_{2n-1}(\mu^e) & p_{2n-2}(\mu^e) & \cdots & p_n(\mu^e) \end{bmatrix}, \quad (34)$$

where $p_i(\mu^e) = 0$ for $i \in \mathbb{Z}^- \cup [n+1, \infty]$. The equivalent conditions for emergence of Hopf bifurcation, stated in *Theorem 3*, are derived as [27]

$$p_n(\mu^e) > 0, \quad \frac{d(\Delta_{n-1}(\mu))}{d\mu} \neq 0,$$

$$\Delta_i(\mu^e) = \det(H_i(\mu^e)) > 0, \quad i \in \{1, 2, \dots, n-2\}, \quad (35)$$

$$\Delta_{n-1}(\mu^e) = \det(H_{n-1}(\mu^e)) = 0.$$

The set of conditions represented in (35), contributes to detection of Hopf bifurcations without the need to solve analytical expressions of eigenvalues.

B. Hopf Bifurcation Control Law

Variations in system parameters could lead to the Hopf bifurcations. Suppose that by these variations, two Hopf bifurcations emerge in stationary branch of the solutions at points (x^{e1}, μ^{e1}) and (x^{e2}, μ^{e2}) , respectively. As it has been mentioned in the introduction section, dynamic state feedback controllers were only able to control *one* Hopf bifurcation [16], [25]. In this study, the controller objective is to eliminate *two* Hopf bifurcations. According to the previous studies, the maximum number of observed Hopf bifurcations in the stationary branch of solutions (before the SNB) of the power systems is two [7]–[9], [20]. A control law is proposed to

modify the nonlinear behavior of the system by eliminating Hopf bifurcations from stationary branch of solutions. The control law u should relocate bifurcations in (x^{e1}, μ^{e1}) and (x^{e2}, μ^{e2}) to $(\bar{x}^{e1}, \bar{\mu}^{e1})$ and $(\bar{x}^{e2}, \bar{\mu}^{e2})$, respectively. This relocation alters the nonlinear behavior of the system. By choosing an appropriate equilibria on locus of the fixed points of the system for relocation of Hopf bifurcations, the bifurcations can be eliminated before the SNB. General feedback control law is formulated as

$$\begin{aligned} u &= u(x, y_c), \\ \dot{y}_c &= h(x, y_c), \end{aligned} \quad (36)$$

where $y_c \in \mathbb{R}^m$ is the m -dimensional ($1 \leq m \leq n$) controller state vector [28]. The control law containing linear and quadratic terms, can be formulated as

$$\begin{aligned} u_i(x_i, y_{c_i}) &= k_{1i}x_i + k_{2i}(x_i - \bar{x}_i^{e1})^2 - l_i y_{c_i}, \\ \dot{y}_c &= u_i(x_i, y_{c_i}). \end{aligned} \quad (37)$$

In (37), \bar{x}_i^{e1} represents the equilibria at the first desired Hopf bifurcation location ($\forall i \in \{1, 2, \dots, m\}$). $K_1 = [k_{11}, k_{12}, \dots, k_{1m}]$ and $K_2 = [k_{21}, k_{22}, \dots, k_{2m}]$ are the control gain vectors and $L = [l_1, l_2, \dots, l_m]$ is the constant parameter vector. Without loss of generality, it can be assumed that m control components are added to the first n state equations of the system. The controlled system can be expressed as

$$\begin{aligned} \dot{x} &= F(x, \mu) + u(x, y_c), \\ \dot{y}_c &= h(x, y_c), \end{aligned} \quad (38)$$

where:

$$\begin{aligned} u(x, y_c) &= [u_1(x_1, y_{c_1}), u_1(x_2, y_{c_2}), \dots, \\ &\quad u_m(x_m, y_{c_m}), 0, \dots, 0]^T, \\ h(x, y_c) &= [u_1(x_1, y_{c_1}), u_1(x_2, y_{c_2}), \dots, u_m(x_m, y_{c_m})]^T. \end{aligned} \quad (39)$$

After adding the controller state equations, the dimension of the controlled system is $n+m$. The controller preserves the equilibria structure of the open-loop system, and the equilibrium of the additional component, which is $y_{c_i}^e = (k_{1i}x_i^e + k_{2i}(x_i^e - \bar{x}_i^{e1})^2)/l_i$, is added to the equilibria structure of the closed-loop.

Theorem 4 (Equilibria Structure Preservation Theorem). *The controller preserves the equilibria structure of the uncontrolled system.*

In addition, the stability of control law is guaranteed for $l_i > 0$, by writing the control law in Laplace domain.

Theorem 5 (Controller Stability Theorem). *the control law is stable for all $l_i > 0$.*

One of the major characteristics of the introduced control law, is that control gain vectors K_1 and K_2 are employed for relocation of Hopf bifurcations independently. In other words, in the controlled system K_1 is used for relocation of one Hopf bifurcation, and it operates independently from K_2 . K_1 is sufficient for control of systems which undergo only one Hopf bifurcation. For systems with two Hopf bifurcations, K_2 is employed after analytical derivation of K_1 and is used for relocation of second Hopf bifurcation.

C. Analytical Calculation of Control Gain Vectors

For implementation of controller, control gain vectors should be calculated. As mentioned above, control gain vector K_1 is independent from K_2 , meaning that for relocation of one Hopf bifurcation there is no need to calculate K_2 . In such case, the Jacobian matrix of the controlled system would only contain the K_1 gain vector resulted from the newly added states to the system (mentioned in (37) and (38)). However, for relocation of two Hopf bifurcations, first K_1 should be calculated and then based on the values of the control gain vector K_1 , the second control gain vector K_2 can be calculated. This is due to the fact that in case of existence of two Hopf bifurcations, the Jacobian matrix of the controlled system should contain both control gain vectors for effective relocation of Hopf bifurcations. In what follows, analytical calculation of control gain vectors are described.

Control gain vectors can be calculated analytically, according to criteria mentioned in (35) and the Jacobian matrix of the controlled system at the desired equilibrium to which Hopf bifurcation has been relocated. Consider the controlled system as

$$\dot{X} = F_c(X, \mu), \quad (40)$$

where

$$\begin{aligned} X &= [x, y_c]^T, \\ F_c &= [F(x, \mu) + u(x, y_c), h(x, y_c)]^T. \end{aligned} \quad (41)$$

The Jacobian matrix of the controlled system, denoted by J_c , can be formulated as [26]:

$$J_c(X, \mu) = \begin{bmatrix} J(x) + A(X) & B(X) \\ C(X) & D(X) \end{bmatrix}, \quad (42)$$

where

$$\begin{aligned} J(x) &= \frac{\partial F(x, \mu)}{\partial x}, A(X) = \frac{\partial u(x, y_c)}{\partial x}, B(X) = \frac{\partial u(x, y_c)}{\partial y_c}, \\ C(X) &= \frac{\partial h(x, y_c)}{\partial x}, D(X) = \frac{\partial h(x, y_c)}{\partial y_c}. \end{aligned} \quad (43)$$

Let $J(x)$ denote the Jacobian matrix of the uncontrolled system. Considering the control law described by (37), $A(X)$, $B(X)$, $C(X)$ and $D(X)$ are derived as

$$\begin{aligned} A(X)_{(n \times n)} &= \begin{bmatrix} R & 0_{(n-m) \times (n-m)} \\ 0_{(n-m) \times (n-m)} & 0_{(m) \times (n-m)} \end{bmatrix}, \\ B(X)_{(n \times m)} &= \begin{bmatrix} S \\ 0_{(n-m) \times (m)} \end{bmatrix}, \\ C(X)_{(m \times n)} &= [R \quad 0_{(m) \times (n-m)}], \quad D(X)_{(m \times m)} = S, \end{aligned} \quad (44)$$

where

$$\begin{aligned} R &= \text{diag}(k_{11} + 2k_{21}(x_1 - \bar{x}_1^{e1}), \dots, k_{1m} + 2k_{2m}(x_m - \bar{x}_m^{e1})), \\ S &= \text{diag}(-l_1, \dots, -l_m). \end{aligned} \quad (45)$$

Hereafter, the analytical calculation of K_1 and K_2 are presented.

1) *Analytical Calculation of K_1* : At $\mu = \bar{\mu}^{e1}$, which is the desired location of first Hopf bifurcation, the equilibrium of the controlled system is $(x, y_c, \mu)|_{e_1} = (\bar{x}^{e1}, k_{1i}\bar{x}_i^{e1}/l_i, \bar{\mu}^{e1})$. The matrix R can be found as

$$R = \text{diag}(k_{11}, \dots, k_{1m}). \quad (46)$$

Hence the Jacobian matrix of the system only depends on K_1 . The characteristic polynomial of Jacobian matrix of the controlled system ($J_c(X)$) at $\mu = \bar{\mu}^{e1}$ is obtained as

$$P(\lambda; \bar{\mu}^{e1}) = p_0(\bar{\mu}^{e1})\lambda^{n+m} + p_1(\bar{\mu}^{e1})\lambda^{n+m-1} + \dots + p_{n+m}(\bar{\mu}^{e1}), \quad (47)$$

Application of criteria in (35), K_1 is obtained by solving (48)

$$\Delta_{n+m-1}(\bar{\mu}^{e1}) = \det(H_{n+m-1}(\bar{\mu}^{e1})) = 0, \quad (48)$$

subject to

$$p_{n+m}(\bar{\mu}^{e1}) > 0,$$

$$\Delta_i(\bar{\mu}^{e1}) = \det(H_i(\bar{\mu}^{e1})) > 0 \quad (i = 1, 2, \dots, n + m - 2),$$

$$\left. \frac{d(\Delta_{n+m-1}(\bar{\mu}^{e1}))}{d\mu} \right|_{\mu=\bar{\mu}^{e1}} \neq 0. \quad (49)$$

After calculation of the gain vector K_1 , control gain vector K_2 can be calculated by a similar procedure.

2) *Analytical Calculation of K_2* : Let K_1^* denoted the calculated values of K_1 from V-C1. At $\mu = \bar{\mu}^{e2}$, which is the desired location of second Hopf bifurcation, the equilibrium of the controlled system is $(x, y_c, \mu)|_{e_2} = (\bar{x}^{e2}, [(k_{1i}^* \bar{x}_i^{e2} + k_{2i}(\bar{x}_i^{e2} - \bar{x}_i^{e1})^2)/l_i], \bar{\mu}^{e2})$. The matrix R is obtained as

$$R = \text{diag}(k_{11}^* + 2k_{21}(\bar{x}_1^{e2} - \bar{x}_1^{e1}), \dots, k_{1m}^* + 2k_{2m}(\bar{x}_m^{e2} - \bar{x}_m^{e1})). \quad (50)$$

Hence the Jacobian matrix of the system only depends on K_2 . The characteristic polynomial of Jacobian matrix of the controlled system ($J_c(X)$) at $\mu = \bar{\mu}^{e1}$ is obtained as

$$P(\lambda; \bar{\mu}^{e2}) = p_0(\bar{\mu}^{e2})\lambda^{n+m} + p_1(\bar{\mu}^{e2})\lambda^{n+m-1} + \dots + p_{n+m}(\bar{\mu}^{e2}), \quad (51)$$

With $K_1 = K_1^*$ obtained from V-C1, K_2 can be obtained by solving (52)

$$\Delta_{n+m-1}(\bar{\mu}^{e2}) = \det(H_{n+m-1}(\bar{\mu}^{e2})) = 0, \quad (52)$$

subject to

$$p_{n+m}(\bar{\mu}^{e2}) > 0,$$

$$\Delta_i(\bar{\mu}^{e2}) = \det(H_i(\bar{\mu}^{e2})) > 0 \quad (i = 1, 2, \dots, n + m - 2),$$

$$\left. \frac{d(\Delta_{n+m-1}(\bar{\mu}^{e2}))}{d\mu} \right|_{\mu=\bar{\mu}^{e2}} \neq 0. \quad (53)$$

For implementation of control law in the introduced power system in Section II, the following remarks are made:

Remark 1. In power systems, SNB is the incipient of voltage collapse mechanism. Hence, no stable equilibrium could be defined for power systems which is located after the occurrence SNB.

Remark 2. In order to eliminate the Hopf bifurcations leading to emergence of stable or unstable limit cycles in the system, the Hopf points are relocated to the stationary branch of solutions located after the SNB point. In other words, without loss of generality, Hopf bifurcations are relocated to the equilibria of the system which are not defined as the stable operating points of power systems. Moreover, the stability of the limit cycles emerged from this bifurcation would not be of importance after relocation, since the system has collapsed at the SNB, which occurs before emergence of limit cycles.

Remark 3. For relocation of Hopf bifurcations, a set of feasible equilibriums on stationary branch of solutions located after SNB have to be obtained. It has been proved that control law does not alter the equilibria structure of the system. However, the stability of the initial operating point of the system obtained from (18), should be preserved as well. Set of feasible equilibriums, denotes a set of points located on the stationary branch of solutions after the SNB, that satisfy this stability condition.

Remark 4. K_1 is sufficient for relocation of Hopf bifurcation when only one Hopf bifurcation occurs in stationary branch of solutions. In such case, K_2 could be disregarded altogether for the sake of simplicity of the designed controller. In such case, the controller contains only a linear state feedback.

Remark 5. By dynamical integration of elements in each bus, and solving the power flow (both of which are dependent on the topology of the network), and following the algorithm proposed for the design of controller, this method can be applied to general power systems.

Remark 6. The control law is a dynamic state-feedback. In systems with high complexity as power systems, not all states are observable. The proposed method for relocation of Hopf bifurcations, in general can be applied to any dynamical system. Further, for application of this method in any dynamics, the observability of the system states should be investigated. It should be noted that in this work, the state-feedback control law is not used for all of the states, but only for one state in each investigated case, which has proven to be effective for relocation of Hopf bifurcations. In the single machine 3-bus system, only the first state of the system, which is the d-axis component of the stator current, is used as the state from which a feed-back is taken. Also for the multi-machine system, the angle of one synchronous generator is considered as the state which is controlled. In other words, there is no assumption that all states are observable. The system is controlled using only the feedback taken from one state, in each case. In general, the proposed control method may be used for the observable states of the system with full functionality. Since the stator current and angle of the synchronous generator are measurable at any time in practice, these states have been utilized for the purpose of controller design in single-machine and multi-machine cases, respectively.

The framework describing the overall process of designing the controller and choosing the feasible equilibria for relocation of Hopf bifurcations, is illustrated with details in Fig. 5.

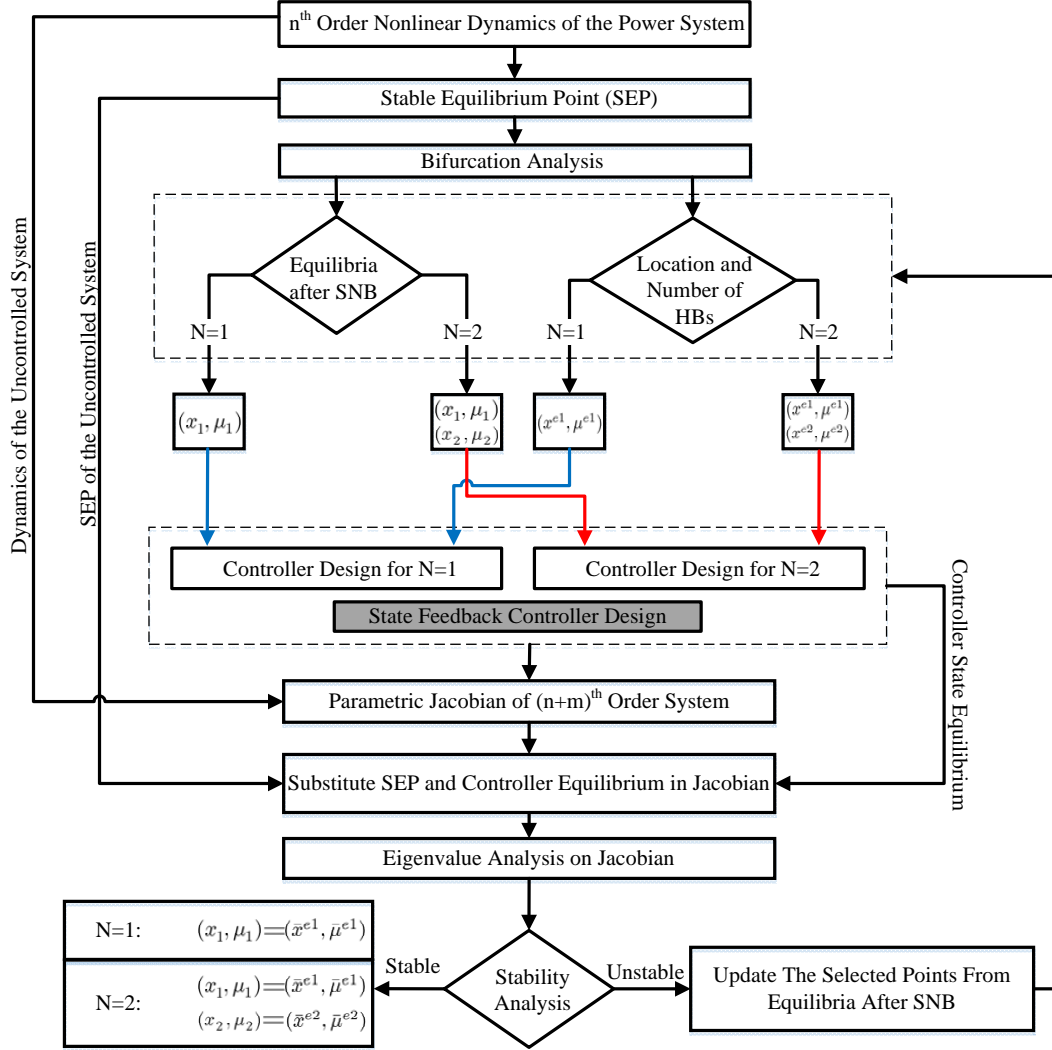


Fig. 5. General framework of the proposed nonlinear state feedback controller for Hopf bifurcation control

D. Controlled System Dynamics

As discussed in Section V-B, state equations of the controller inputs are added to the system. Hence, the dimensions of the controlled system is increased by the number of the controller of states. Considering the dimensions of the power system's dynamics to be n , and the dimensions of the controller to be m , the dimension of controlled power system becomes $n+m$, and the dynamics and states of the single-machine system would be,

$$\begin{aligned} \dot{X} &= F_c(X, \mu), \\ X &= [i_d, i_{mf}, i_{D1}, i_q, i_{Q1}, \omega, \delta, v_l, \theta_l, y_{c1}, \dots, y_{cm}]^T. \end{aligned} \quad (54)$$

And for the case of multi-machine dynamics,

$$\begin{aligned} \dot{X} &= F_c(X, \mu), \\ X &= [\delta_1, \omega_1, E'_{q1}, \delta_2, \omega_2, \delta_3, \omega_3, y_{c1}, \dots, y_{cm}]^T. \end{aligned} \quad (55)$$

Also, in differential equations mentioned above, free parameters used for bifurcation analysis of the single-machine system can be represented in terms of P_{ld} and Q_{ld} as

$$\mu = [P_{ld}, Q_{ld}]^T. \quad (56)$$

And for the multi-machine system,

$$\mu = \lambda. \quad (57)$$

VI. CONCLUSION

In this paper, saturation phenomenon of synchronous generator has been integrated into nonlinear dynamics of power systems. The purpose of such integration is demonstration of rich and realistic nonlinear behavior of system. For inclusion of saturation, an analytical model has been used which is accurate in modeling the subtransient, transient and steady states of system dynamics. The chosen method for modeling

saturation proves to be robust against variations of equilibria, which is an important factor for the purposes of bifurcation analysis. In addition, a dynamic state-feedback control law has been proposed which relocates the Hopf bifurcations occurring before the Saddle-Node bifurcation, to stationary solutions which are located after the Saddle-Node bifurcation. Hence, the control law guarantees the stability of the system before SNB, eliminates the possibility of voltage collapse before SNB, and an undesired oscillatory behavior from system responses. After obtaining the nonlinear high dimensional dynamics of the power system, bifurcation analysis will be performed with respect to active and reactive power demands of the nonlinear load. The second part of this paper [19], validates the performance of the proposed controller in elimination of Hopf bifurcations.

ACKNOWLEDGMENT

The authors would like to acknowledge the valuable inputs, fruitful comments, and discussions of Prof. Konstantin Turitsyn from Massachusetts Institute of Technology to this article. The quality of this work is substantially improved by his comments.

APPENDIX A

PROOF OF IMPLICIT FUNCTION THEOREM

In this appendix, the proof for *Theorem 1* is provided. To this end, first two preliminary Lemmas which are substantially required before proceeding with the main proof, are introduced.

Lemma 1. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable, $T : \mathbb{R}^k \rightarrow \mathbb{R}^m$ be the linear function associated to a $n \times k$ real matrix M , and y be a fixed point in \mathbb{R}^n . Then, the function $G(x) = F(y + Tx), \forall x \in \mathbb{R}^k$, is differentiable and satisfies $DG(x) = DF(y + Tx)M, \forall x \in \mathbb{R}^k$.

Lemma 2. Let F be in $C^1(\Omega; \mathbb{R}^n)$, and $p \in \Omega$ satisfying $\text{Det}(DF(p)) \neq 0$. Then F restricted to some ball $B(p; r)$ with $r > 0$, is injective.

Proof of Lemmas 1 and 2 are given in [29].

Proof (Implicit Function Theorem). Defining $\Psi(x, y) = (x, F(x, y)), (x, y) \in \Omega$, will lead to:

$$\text{Det}(D\Psi) = \text{Det}\left(\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \end{bmatrix}\right) = \text{Det}\left(\frac{\partial F}{\partial y}\right). \quad (58)$$

where \mathbf{I} is the identity matrix of order n and $\mathbf{0}$ the $n \times m$ zero matrix. Thus, according to *Lemma 2* it is concluded that Ψ is injective and $\Omega = \mathcal{H}' \times \mathcal{Y}$, with \mathcal{H}' representing an open set in \mathbb{R}^n containing a , and \mathcal{Y} showing an open set in \mathbb{R}^m containing b . It is proven that $F(x, f(x)) = 0$ subject to $f(a) = b$ has a solution $f = f(x)$ of class C^1 on some open set containing a [29]. Differentiating $F(x, f(x)) = 0$ as

$$\frac{\partial F_i}{\partial x_k} + \sum_{j=1}^m \frac{\partial F_i}{\partial y_j} \frac{f_j}{\partial x_k} = 0. \quad (59)$$

where $1 \leq i \leq m$ and $1 \leq k \leq n$. Rewriting (59) in matricial form gives

$$\frac{\partial F}{\partial x}(x, f(x)) + \frac{\partial F}{\partial y}(x, f(x))Df(x) = 0. \quad (60)$$

Let $g(\cdot)$ satisfy $F(x, g(x)) = 0, \forall x \in \mathcal{H}$, and $g(a) = b$. Hence, $\Psi(x, g(x)) = (x, F(x, g(x))) = (x, 0) = \Psi(x, f(x)), \forall x \in \mathcal{H}$. The injectivity of Ψ implies that $g(x) = f(x), \forall x \in \mathcal{H}$. ■

APPENDIX B

PROOF OF BIFURCATION THEOREMS

In this appendix, the proofs for *Theorem 2* and *Theorem 3* are provided. The preliminaries required for the presented proofs, such as *Lyapunov-Schmidt reduction* are given in [30].

Proof (Saddle-Node Bifurcation Theorem). By a translation in state space \mathbb{R}^n and parameter space, we assume $p = 0$ and $\mu_0 = 0$. From $\langle w, D_x F(0, 0) \rangle = \langle D_x F(0, 0)^T w, u \rangle = 0$. It is observed that w is perpendicular to the image $\text{Im } D_x f(0, 0)$. Thus, the second condition of *Theorem 2* gives

$$\text{Im } D_x f(0, 0) \oplus D_\mu f(0, 0) = \mathbb{R}^n. \quad (61)$$

Also, the third condition of *Theorem 2* gives

$$D_{xx} f(0, 0)(v, v) \notin \text{Im } D_x f(0, 0). \quad (62)$$

Hence,

$$D_x f(0, 0)x - N(x, \mu) = 0, \quad (63)$$

where $N(x, \mu) = f(x, \mu) - D_x f(x, \mu)$. Application of Lyapunov-Schmidt reduction to (63) yields,

$$(I - E)N(y_z^*(y, \mu), \mu) = 0, \quad (64)$$

where z^* solves,

$$z - KEN(y + z, \mu) = 0. \quad (65)$$

According to (65), it can be obtained

$$D_y z^*(0, 0) - KED_x N(0, 0)(I + D_y z^*(0, 0)) = 0. \quad (66)$$

Hence, $D_y z^*(0, 0) = 0$. Further,

$$\begin{aligned} & D_\mu((I - E)N(y + z^*(y, \mu), \mu))|_{y=0, \mu=0} \\ &= (I - E)(D_x N(0, 0)D_\mu z^*(0, 0) + D_\mu N(0, 0)) \\ &= (I - E)D_\mu N(0, 0) \\ &= \langle w, D_\mu N(0, 0) \rangle. \end{aligned} \quad (67)$$

which is nonzero by assumption. Considering $y = \eta v$ with $\eta \in \mathbb{R}$,

$$\begin{aligned} & D_\eta N(\eta v + z^*(\eta v, \mu), \mu) \\ &= D_x N(\eta v + z^*(\eta v, \mu), \mu)(v + D_y z^*(\eta v, \mu)v), \end{aligned} \quad (68)$$

By differentiating again with $D_x N(0, 0) = 0$ and $D_y z^*(0, 0) = 0$,

$$\begin{aligned} & D_{\eta\eta}((I - E)N(\eta v + z^*(\eta v, \mu), \mu))|_{\eta=0, \mu=0} \\ &= (I - E)(D_{xx} N(0, 0)(v, v)) \\ &= \langle w, D_{xx} N(0, 0)(v, v) \rangle. \end{aligned} \quad (69)$$

which is nonzero by assumption. It is shown that the reduced bifurcation equation satisfies the assumptions of saddle-node bifurcation in one dimension, and the proof is concluded. ■

Proof (Hopf Bifurcation Theorem). By linearization at the origin, the differential equations become

$$\begin{aligned}\dot{x} &= \alpha(\mu)x - \beta(\mu)y + g(x, y, \mu), \\ \dot{y} &= \beta(\mu)x + \alpha(\mu)y + h(x, y, \mu).\end{aligned}\quad (70)$$

where g , h and their first order derivatives with respect to (x, y) disappear at the origin. Note that $\alpha(0) = 0$, $\beta(0) \neq 0$ and by assumption $\alpha'(0) \neq 0$. It is convenient to use polar coordinates for the analysis purpose. By transformation of the equations to polar coordinates the following is obtained

$$\begin{aligned}\dot{r} &= \alpha(\mu)r + p(r, \theta, \mu), \\ \dot{\theta} &= \beta(\mu)r + q(r, \theta, \mu).\end{aligned}\quad (71)$$

where

$$\begin{aligned}p(r, \theta, \mu) &= g(r \cos(\theta), r \sin(\theta), \mu) \cos(\theta) \\ &\quad + h(r \cos(\theta), r \sin(\theta), \mu) \sin(\theta), \\ q(r, \theta, \mu) &= \frac{1}{r}(h(r \cos(\theta), r \sin(\theta), \mu) \cos(\theta) \\ &\quad - g(r \cos(\theta), r \sin(\theta), \mu) \sin(\theta)).\end{aligned}\quad (72)$$

Since g , h and their first order derivatives vanish at the origin, the system is smooth. The equilibrium at the origin has been blown up to a circle $\{0\} \times \mathbb{R} \times \mathbb{T}$. To proceed with the proof, Poincaré first return map on $\mathbb{R} \times \{0\}$ is studied [23]. By normal form transformation of equations, the following is derived,

$$\begin{aligned}\dot{r} &= \alpha r + \operatorname{Re}(c_1)r^3 + \mathcal{O}(r^4), \\ \dot{\theta} &= \beta - \operatorname{Im}(c_1)r^2 + \mathcal{O}(r^3).\end{aligned}\quad (73)$$

By time reparametrization,

$$\begin{aligned}\dot{r} &= \frac{\alpha}{\beta}r - ar^3 + \mathcal{O}(r^4), \\ \dot{r} &= 1.\end{aligned}\quad (74)$$

with $a = \operatorname{Re}(c_1) - (\frac{\alpha}{\beta^2})\operatorname{Im}(c_1)$ a smooth function of the parameter. For $\mu = 0$, $a = \operatorname{Re}(c_1)$. It should be noted that higher order terms are functions of r , θ and the parameter μ . More detailed analysis concludes that a is in fact the first Lyapunov coefficient. To find the periodic orbit, r is rescaled to make the coefficient of the third order term equal to ± 1 . Considering the specific case where $c_1 > 0$,

$$\dot{\rho} = \frac{\alpha}{\beta}\rho - \rho^3 + \mathcal{O}(|\rho|^4).\quad (75)$$

Thus, ρ is defined as a function of θ . Writing the Taylor expansion for the solution starting with ρ_0 yields,

$$\rho(\theta, \rho_0) = u_1(\theta)\rho_0 + u_2(\theta)\rho_0^2 + u_3(\theta)\rho_0^3 + \mathcal{O}(|\rho|^4).\quad (76)$$

Considering (75) and (76) gives,

$$\begin{aligned}(\frac{\alpha}{\beta})u_1\rho_0 + (\frac{\alpha}{\beta})u_2\rho_0^2 + ((\frac{\alpha}{\beta})u_3 - u_1^3)\rho_0^3 + \mathcal{O}(|\rho|^4) \\ = u_1'\rho_0 + u_2'\rho_0^2 + u_3'\rho_0^3 + \mathcal{O}(|\rho|^4).\end{aligned}\quad (77)$$

which gives,

$$\begin{aligned}u_1' &= (\frac{\alpha}{\beta})u_1, \\ u_2' &= (\frac{\alpha}{\beta})u_2, \\ u_3' &= (\frac{\alpha}{\beta})u_3 - u_1^3.\end{aligned}\quad (78)$$

Solving (78) for $u_1(0) = 1$, $u_2(0) = 0$ and $u_3(0) = 0$ results in,

$$\begin{aligned}u_1(\theta) &= e^{(\frac{\alpha}{\beta})\theta}, \\ u_2(\theta) &= 0, \\ u_3(\theta) &= e^{(\frac{\alpha}{\beta})\theta} \frac{1 - e^{-2(\frac{\alpha}{\beta})\theta}}{2(\frac{\alpha}{\beta})}.\end{aligned}\quad (79)$$

Thus, the return map $\rho_0 \rightarrow \rho(2\pi, \rho_0)$ is obtained as

$$\rho(2\pi, \rho_0) = e^{2\pi(\frac{\alpha}{\beta})}\rho_0 - e^{2\pi(\frac{\alpha}{\beta})}(2\pi + \mathcal{O}(\alpha))\rho_0^3 + \mathcal{O}(|\rho_0|^4).\quad (80)$$

By analysis of the map obtained in (80), a Hopf bifurcation is found. This concludes the proof. ■

APPENDIX C

PROOF OF EQUILIBRIA STRUCTURE PRESERVATION AND CONTROL LAW STABILITY THEOREMS

In this appendix, the proof for *Theorem 4* and *Theorem 5* is provided.

Proof (Equilibria Structure Preservation Theorem) The equilibria of the system is derived by time derivatives to zero.

$$\begin{aligned}0 &= f(x, \mu) + u(x, y), \\ 0 &= g(x, y).\end{aligned}\quad (81)$$

According to (39), it is concluded that for (81) to hold,

$$0 = u(x, y).\quad (82)$$

Thus, the equilibria of the uncontrolled system are preserved in the controlled system. ■

Proof (Equilibria Control Law Stability) Let u_i denote the output, and the input z_i is defined as

$$z_i = k_{1i}x_i + k_{2i}(x_i - \bar{x}_i^{e1})^2.\quad (83)$$

Hence, the equations in Laplace domain are as

$$\begin{aligned}u_i(s) &= z_i(s) - l_i y_i(s), \\ (s + l_i)y_i(s) &= z_i(s).\end{aligned}\quad (84)$$

The transfer function G_i is obtained as

$$G_i(s) = \frac{u_i(s)}{z_i(s)} = \frac{s}{s + l_i}.\quad (85)$$

Since for $l_i > 0$ the system the poles of transfer function $G_i(s)$ are located at the left side of the s -plane, the stability of the control law can be guaranteed for $l_i > 0$ [16]. ■

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