Linear Ringdown Analysis Methods

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***Abstract*—This paper provides a summary of linear ringdown analysis methods excerpted from the first chapter of the June 2012 IEEE Power & Energy Society special publication entitled "Identification of Electromechanical Modes in Power Systems." This publication is a report of the Task Force on Identification of Electromechanical Modes of the Power System Stability Subcommittee of the Power System Dynamic Performance Committee [1]. Chapter 1 of this special publication provides a basic summary of three linear methods: Prony, eigensystem realization methods (ERA), and matrix pencil methods, followed by detailed applications. This paper provides an overview of Chapter 1.**

Index Terms- **ringdown analysis, Prony methods, eigensystem realization methods, matrix pencil methods, electromechanical modal identification**

I. Introduction

Although many systems are inherently nonlinear, in some instances they may respond to well-tuned linear controls. In order to implement linear feedback control, the system designer must have an accurate model of sufficiently low order from which to design the control. Several approaches to developing such lower-order models have included dynamic equivalencing, eigenanalysis, and pole/zero cancellation. Frequently, however, the original system is too complex or the parameters are not known with enough accuracy to produce an adequate reduced order model. In practice, the system may have parameters that drift with time or operating condition which compromises the accuracy of the mathematical model. In these cases, it is desirable to extract the modal information directly from the system response to a perturbation. Using this approach, it may be possible to replace the actual dynamic model with an estimated linear model that is derived from the system response to a stimulus. The time-varying dynamic response of a power system to a disturbance may be composed of numerous modes that must be identified. Several methods have been proposed to extract the pertinent modal information from time varying responses. The application of an appropriate identification method must recognize the system nonlinearities, the size of the model that can be effectively utilized, and the reliability of the results.

Methods that are applied directly to the nonlinear system simulation or field measurements include the effects of nonlinearities. In full state eigenvalue analysis, the size of the system model is typically limited to several hundred states with present computing capabilities. This means that a typical system containing several thousand nodes must be reduced using dynamic equivalencing. Modal analysis techniques that operate directly on system output are not limited by system size. This means that standard time-domain-analysis results are directly applicable. This eliminates the possibility of losing some of system modal content due to reduction. The estimated linear model may then be used for control design applications or other linear analysis techniques. The estimated model is typically of lower order than the original system, but still retains the dominant modal characteristics.

II. Overview of Methods

The modal analysis problem may be posed, such that given a set of measurements that vary with time, it is desired to fit a time-varying waveform of pre-specified form to the actual waveform (i.e., minimize the error between the actual measured waveform and the proposed waveform). The coefficients of the pre-specified waveform yield the dominant modal characteristics of the underlying linear system. Consider the following linear system:

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0 \tag{1}$$

$$y = Cx + Du \tag{2}$$

where $\dot{x}$ denotes differentiation of x with respect to time. Variables u and y are respectively the input and the output of the system; x, the internal state of the system, is usually taken to be a vector of n elements (n being the order of the system differential equation). These equations, and the system matrices within them, can be rearranged in many different ways to serve specific purposes. Each individual element x_i can be given by:

$$x_i(t) = \sum_{i=1}^{n} r_i x_{i0} e^{\lambda_i t} = \sum_{i=1}^{n} a_i e^{\sigma_i t} \cos\left(\omega_i t + \theta_i\right) \qquad (3)$$

The parameter r_i is the residue of the mode i, x_{i0} is derived from influence of the initial conditions, and λ_i represents the (possibly complex) eigenvalues of A. The estimation of these responses yields modal information about the system that can be used to predict possible unstable behavior, controller design, parametric summaries for damping studies, and modal interaction information.

The discrete form of equations (1) and (2) is given by:

$$x(k+1) = Ax(k) + Bu(k) \qquad (4)$$

$$y(k) = Cx(k) + Du(k) \qquad (5)$$

where k represents the discrete time interval. This system of equations is shown in Figure 1.

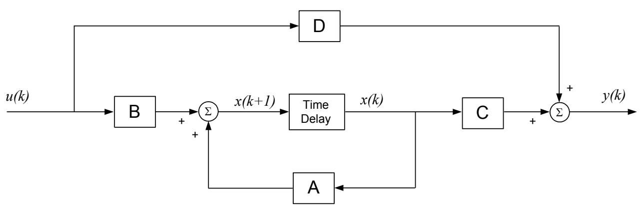

Figure 1: Discrete Linear System Representation

The primary task in modal identification is to determine the system poles or, equivalently, the eigenvalues of A. Transfer function identification must, in addition to the poles, also determine the zeros and the gains along one or more response paths. The system transfer function involves all of the system matrices in equations (1) and (2):

$$\begin{aligned} T(s) &= \frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)} \\ &= \frac{G\left(s^m + a_{m-1}s^{m-1} + \ldots + a_1 s + a_0\right)}{\left(s^n + b_{m-1}s^{n-1} + \ldots + b_1 s + b_0\right)} \\ &= \frac{G(s-z_1)(s-z_2)\ldots(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_n)} \\ &= \sum_{i=1} \frac{K_i}{s-p_i} \end{aligned} \qquad (6)$$

where each pole p_i is identical to an eigenvalue λ_i of the matrix A and K_i is the transfer function residue of the associated pole p_i.

Ringdown analysis, which is loosely correlated to the impulse response of the system, is based upon modal decompositions of output vector *y(t)*. The modes and the modal parameters identified are for a subset of A that is estimated from a subset of *y(t)*; the mode shape parameters are specific to whatever stimulus may have produced the output. Given sufficient knowledge of *u(t)*, an approximating subset can be constructed for the matrices of equations (1) and (2).

Any time-varying function, such as a ringdown trace, can be fit to a series of complex exponential functions over a finite time interval. However, in most systems, a true fit is not possible because

- the system response may not be truly linear
- only a subset of the modal content is excited by the event

However, it is not practical to include a large number of terms in the fitting function. The problem then becomes one of minimizing the error between the actual time-varying function and the proposed function by estimating the magnitude, phase, and damping parameters of the fitting function. Several methods have been proposed for transfer function identification and modal analysis. The three primary methods that have been developed are:

- Prony
- Eigensystem Realization Algorithm (ERA)
- Matrix Pencil

All of these methods are approximate, and none can generate results of higher quality than the information provided to them. Results from model-based eigenanalysis are colored by errors in the model, and by linear approximations to nonlinear phenomena such as saturation and dead zones. Results from measurement-based eigenanalysis are colored by the extent and quality of the available signals. Some modes may not be sufficiently observable within the signal set. Those which are observable may be obscured by noise, by dynamic nonlinearities, and by hidden inputs to the system.

Each of these methods will be briefly summarized in this paper. Greater detail can be found in the report [1].

III. PRONY METHODS

The core notion in Prony analysis originated in an earlier century [2]. Its practical use was not possible until the advent of the digital computer and means for dealing with some inherently ill-conditioned numeric were developed. Prony methods and their modern extensions are designed to directly estimate the parameters for the exponential terms in (3), by fitting a function to an observed record for *y(t).* In doing this it may also be necessary to model offsets, trends, noise, and other extraneous effects in the signal. The Prony method is a "polynomial" method in that it includes the process of finding the roots of a characteristic polynomial.

Let the record for $y(t)$ consist of N samples $y(t_k)$ that are evenly spaced by an amount Δt. The notation is simplified if (3) is recast in the exponential form:

$$\hat{y}(t) = \sum_{i=1}^{n} A_i e^{\sigma_i t} \cos(\omega_i t + \theta_i) \tag{7}$$

where $n \le N$ is the subset of modes to be determined. At the sample times t_k, this can be can be discretized to

$$\hat{y}(k) = \sum_{i=1}^{n} B_i z_i^k \tag{8}$$

where

$$z_i = \exp(\lambda_i \Delta t) \tag{9}$$

The z_i are the roots of the polynomial

$$z^n - \left(a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_{n-1} z^0\right) = 0 \tag{9}$$

where the a_i coefficients are unknown and are calculated from the set of measurement vectors.

The strategy for obtaining a Prony solution can be summarized as follows:

Step 1: Assemble selected elements of the record into a Toeplitz data matrix

Step 2: Fit the data with a discrete linear prediction model, such as a least squares solution.

Step 3: Find the roots of the characteristic polynomial (9) associated with the model of step 1.

Step 4: Using the roots of step 3 as the complex modal frequencies for the signal, determine the amplitude and initial phase for each mode.

The approach to the Toeplitz (or the closely related Hankel) matrix assembly of Step 1 has received the most attention in the literature. The problem can be formulated in many different ways. If the initial (i.e., $i < 0$) and post (i.e., $i > N$) conditions are assumed to be zero, then the subscript ranges X_1 through X_4 in equation (7) represent four such formulations. The range X_1 is termed the covariance problem. Because the initial and post conditions are not used, no assumption is required on their value. The range X_4 is called the correlation problem; it incorporates both initial and post conditions. The remaining problems, X_2 and X_3, are termed the pre-windowed and post-windowed methods.

In the majority of practical cases, the Toeplitz matrix is non-square with more rows than columns. A Toeplitz matrix is a matrix with a constant diagonal in which each descending diagonal from left to right is constant. The system of equations **Error! Reference source not found.**) requires a least squares solution to find the factors a_1 through a_n. After the a_i coefficients are obtained, the n roots z_i of the polynomial in (9) can be found by factoring.

$$X_4 \quad X_3 \quad X_2 \quad X_1 \quad \begin{bmatrix} y_0 & 0 & \cdots & 0 \\ y_1 & y_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_{n-1} & \cdots & y_0 \\ y_{n+1} & y_n & \cdots & y_1 \\ \vdots & \vdots & \ddots & \vdots \\ y_N & y_{N-1} & \cdots & y_{N-n-1} \\ 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & y_N \end{bmatrix} \begin{bmatrix} -1 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{11}$$

Step 4 is also a linear algebra problem. Once the roots z_i are obtained from step 3, they are substituted into (8) and written in matrix form as

$$\begin{bmatrix} z_1^0 & z_2^0 & \ldots & z_n^0 \\ z_1^1 & z_2^1 & \ldots & z_n^1 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ z_1^{N-1} & z_2^{N-1} & \ldots & z_n^{N-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ . \\ . \\ . \\ B_{N-1} \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ . \\ . \\ . \\ y(N-1) \end{bmatrix} \tag{10}$$

The $N \times n$ matrix in (12) is a Vandermonde matrix and solving it for the B_i is called the Vandermonde problem. Once the residue coefficients are found, the estimated signal $\hat{y}(t)$ can be reconstructed from (8) using the roots of (9). The reconstructed signal $\hat{y}(t)$ will usually not fit $y(t)$ exactly. An appropriate measure for the quality of this fit is the signal to noise (SNR) ratio:

$$\mathrm{SNR} = 20 \log \frac{\|\hat{y} - y\|}{\|y\|} \tag{11}$$

where the SNR is given in decibels (dB).

At a more abstract level, Prony analysis is sometimes characterized as a projection method in which steps 1 and 2 define a modal basis onto which the observed data $y(k)$ are projected at step 3. Once this basis is determined, some or all of it can be reused in subsequent "repeat" solutions. Conditions under which this is useful include the following:

- mode shapes are desired for a larger number of signals than can be processed in one tandem analysis

- modal residues are desired at locations where the signal to noise ratio is adverse
- the signal to be analyzed consists of a fast transient imposed upon a much slower one

The first step in implementation is to model the trend, perhaps on a time frame ranging from 3 seconds to 30 seconds. Then, retaining this estimate, another repeat solution would be performed from 3 seconds to about 15 seconds. This two-step approach permits better separation of the trend from the swing dynamics, while avoiding inclusion of the "noise tail" in the final estimate of modal parameters.

IV. Eigensystem Realization Algorithm

The Eigensystem Realization Algorithm (ERA) is based on the singular value decomposition of the Hankel matrix H_0 associated with the linear ringdown of the system. The ERA fits a discrete state space model to the impulse response of a linear system. From the state space model, modal frequencies and damping coefficients can be calculated. Succinctly, the algorithm consists of building a Hankel matrix whose elements are the Markov parameters of the system under study; from this matrix, the state space matrices are derived using the singular value decomposition (SVD) of the Hankel matrix. Lastly, the system modes are computed from the realized system matrices. Figure 2 is a conceptual view of the ERA identification process.

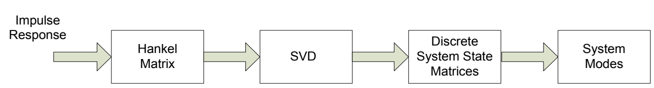

Figure 2: ERA Process

A Hankel matrix is a square matrix with constant skew-diagonals. The Hankel matrices are typically assembled using all of the available data such that the top left-most element of H_0 is y_0 and the bottom right-most element of H_1 is y_N. The Hankel matrices are assembled such that:

$$H_0 = \begin{bmatrix} y_0 & y_1 & \cdots & y_r \\ y_1 & y_2 & \cdots & y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_r & y_{r+1} & \cdots & y_{N-1} \end{bmatrix} \qquad (12)$$

$$H_1 = \begin{bmatrix} y_1 & y_2 & \cdots & y_{r+1} \\ y_2 & y_3 & \cdots & y_{r+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_{r+1} & y_{r+2} & \cdots & y_N \end{bmatrix} \qquad (13)$$

and r is $\frac{N}{2}-1$. This choice of r assumes that the number of data points is sufficient such that $r > n$.

The ERA formulation begins by separating the singular value decomposition of H_0, into two components according to the relative size of the singular values:

$$H_0 = U \Sigma V^T = \begin{bmatrix} U_n & U_z \end{bmatrix} \begin{bmatrix} \Sigma_n & 0 \\ 0 & \Sigma_z \end{bmatrix} \begin{bmatrix} V_n^T \\ V_z^T \end{bmatrix} \qquad (14)$$

where Σ_n and Σ_z are diagonal matrices with their elements ordered by magnitude:

$$\Sigma_n = \text{diag}\left(\sigma_1, \sigma_2, \ldots, \sigma_n\right)$$

$$\Sigma_z = \text{diag}\left(\sigma_{n+1}, \sigma_{n+2}, \ldots, \sigma_N\right)$$

and the singular values are ordered by magnitude such that

$$\sigma_1 > \sigma_2 > \ldots > \sigma_n > \sigma_{n+1} > \sigma_{n+2} > \ldots > \sigma_N$$

The SVD is a useful tool for determining an appropriate value for n. The ratio of the singular values of contained in Σ can determine the best approximation of n. The ratio of each singular value σ_i to the largest singular value $\sigma_{\max}$ is compared to a threshold value, where p is the number of significant decimal digits in the data:

$$\frac{\sigma_i}{\sigma_{\max}} \approx 10^{-p}$$

An example is to set p equal to 3 significant digits, thus any singular values with a ratio below 10^{-3} are assumed to be part of the noise and are not included in the reconstruction of the system. The value of n should be set to the number of singular values with a ratio above the threshold 10^{-p}. It can be shown that for a linear system of order n, the diagonal elements of Σ_z are zero (assuming that the impulse response is free of noise). The practical significance of this result is that the relative size of the singular values provides an indication of the identified system order. If the singular values exhibit a significant grouping such that, $\sigma_n >> \sigma_{n+1}$ then from the partitioned representation given in (14), H_0 can be approximated by

$$H_0 \approx U_n \Sigma_n V_n^T \quad (17)$$

The method for obtaining the eigenvalue realization algorithm solution can be summarized as follows:

Step 1: Assemble selected elements of the record into a Hankel data matrices H_0 and H_1

Step 2: Perform the singular value decomposition of H_0 and estimate the system order n based on the magnitude of the singular values

Step 3: Compute the discrete system matrices as follows:

$$A = \Sigma_n^{-1/2} U_n^T H_1 V_n \Sigma_n^{-1/2}$$
$$B = \Sigma_n^{1/2} V_n^T (1:n, 1:n_u)$$
$$C = U_n (1:N, 1:n) \Sigma_n^{1/2}$$
$$D = y_0$$

Step 4: Calculate continuous system matrices A_c, B_c assuming a zero order hold and sampling interval Δt:

$$A_c = \ln\left(\frac{A}{\Delta t}\right)$$

$$B_c = \left[\int_0^{\Delta t} e^{A\tau} d\tau\right]^{-1} B$$

The reduced system response can then be computed from the continuous matrices.

V. MATRIX PENCIL METHOD

The Matrix Pencil approach was introduced for extracting poles from antennas' electromagnetic transient responses. The matrix pencil method produces a matrix whose roots provide z_i. The poles are found as the solution of a generalized eigenvalue problem.

The advantage of the Matrix Pencil method is that the signal poles can be found directly from the eigenvalues of a single developed matrix in contrast to polynomial methods which require a two-step process. This method is designed to directly estimate the parameters for the exponential terms by fitting the function (3) to an observed measurement for *y(t)* in (7), where *y(t)* consists of N samples that are evenly spaced by a time interval Δt. Since the measurement signal *y(t)* may contain noise or dc offset, it may have to be conditioned before the fitting process is applied.

By using the generalized eigenvalue solution to find z_i, the Matrix Pencil method removes the limitation on the number of poles M, whereas, the polynomial method has difficulties obtaining roots of a polynomial if M is greater than 50. This results in the estimates of z_i having better statistical properties.

The basic process of the Matrix Pencil is similar to that of the ERA method up through Step 3:

Step 1: Assemble selected elements of the record into a Hankel data matrix

Step 2: Fit the data with a discrete linear prediction model, such as a least squares solution.

Step 3: Define the matrices V_1 and V_2 from V in (14):

$$[V_1] = [v_1 \; v_2 \; v_3 \ldots v_{n-1}]$$
$$[V_2] = [v_2 \; v_3 \; v_4 \ldots v_n]$$

and calculate the matrices Y_1 and Y_2:

$$[Y_1] = [V_1]^T [V_1]$$
$$[Y_2] = [V_2]^T [V_1]$$

Step 4: The desired poles z_i may be found as the generalized eigenvalues of the matrix pair $\{[Y_2] - \lambda [Y_1]\}$. The eigenvalue set $\lambda(Y_2, Y_1)$ is contained in the square matrices Y_1 and Y_2, as the pencil values or roots of Y_2 relative to Y_1.

VI. CONCLUSIONS

This paper provides a brief summary of the first chapter of the IEEE special publication report.

REFERENCES

[1] IEEE Publications, "Identification of Electromechanical Modes in Power Systems," IEEE Power & Energy Society, Special publication TP462, June 2012.

[2] IEEE Publications, "Eigenanalysis and Frequency Domain Methods for System Dynamic Performance," IEEE Power Engineering Society, 1990.