Converting Level Set Gradients
to Shape Gradients

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Abstract. The level set representation of shapes is useful for shape evolution and is widely used for the minimization of energies with respect to shapes. Many algorithms consider energies depending explicitly on the signed distance function (SDF) associated with a shape, and differentiate these energies with respect to the SDF directly in order to make the level set representation evolve. This framework is known as the "variational level set method". We show that this gradient computation is actually mathematically incorrect, and can lead to undesirable performance in practice. Instead, we derive the expression of the gradient with respect to the shape, and show that it can be easily computed from the gradient of the energy with respect to the SDF. We discuss some problematic gradients from the literature, show how they can easily be fixed, and provide experimental comparisons illustrating the improvement.

1 Introduction

In recent years, much work on geometric active contour models, i.e. active contour models [1] implemented with the level set method [2], has been proposed to solve many computer vision problems [3]. Any planar closed curve Γ, i.e. any function $\Gamma : \mathbb{S}^1 \to \Omega$ from the circle $\mathbb{S}^1$ to the image domain $\Omega \subset \mathbb{R}^2$, can be represented by the zero level set of a higher-dimensional embedding function $\phi : \Omega \to \mathbb{R}$. During curve evolution, instead of directly updating the contour Γ, one can then update its associated embedding function ϕ, which is more practical for handling topological changes, like merging or splitting. The embedding function ϕ is usually the signed distance function (SDF) of Γ, i.e. the function that associates any point $x \in \Omega$ with the signed distance $\phi(x) = \pm d(x, \Gamma)$ from x to Γ, with a minus sign if x belongs to the interior of the region delimited by Γ.

Many computer vision problems involving shapes can be formulated as the minimization of a certain energy functional $E(\Gamma)$. Depending on the properties of the energy E, one is often reduced to performing gradient descents with respect

to the shape Γ, starting from an initialization Γ_0 and making Γ evolve step by step in the opposite direction of the gradient of the energy:

$$\begin{cases} \Gamma(0) & = \ \Gamma_0 \\ \dfrac{\partial \Gamma(t)}{\partial t} & = \ -\nabla_\Gamma E(\Gamma(t)) \end{cases} \tag{1}$$

where the gradient $\nabla_\Gamma E(\Gamma)$ is defined from the derivative of E with respect to Γ and depends on the choice of an inner product (see Section 3.1 for a proper definition, or [4]). If one chooses to represent the contour $\Gamma(t)$ by an embedding function $\phi(t)$, then one is interested in the equation that governs the evolution of $\phi(t)$. Since by definition $\forall t, \ \forall x \in \Gamma(t), \quad \phi(t)(x) = 0$, one obtains, by differentiation, $\frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \Gamma}{\partial t} = 0$, and thus:

$$\begin{cases} \phi(0) & = \ \phi_0 \quad (\text{e.g.} := \ \mathrm{SDF}(\Gamma_0) \ ) \\ \dfrac{\partial \phi}{\partial t}(x) & = \ |\nabla_x \phi(t)|(x) \ V(t)(x) \qquad \forall x \in \Gamma(t) = \phi(t)^{-1}(0) \end{cases} \tag{2}$$

where $\forall x \in \Gamma, \ V(t)(x) = \nabla_\Gamma E(\Gamma(t))(x) \cdot \mathbf{n}_{\Gamma(t)}(x)$ is the normal velocity field, i.e. the part of the shape gradient that is normal to the contour Γ. This so-called "level set equation" has to be properly defined for points which are not on Γ, e.g. by extending the velocity field V to at least a narrow band around Γ.

However, for many applications, calculating the gradient using Eq.(1) directly is difficult and therefore an alternative derivation to obtain the level-set equation was proposed in [5], named the "variational level set method". The idea is that since there is a bijection between Γ and its signed distance function ϕ, an energy defined on shapes $E(\Gamma)$ can be rewritten as an energy $F(\phi)$ defined on their level set representations and vice versa. Subsequently, one might be interested in deriving the Euler-Lagrange equation that minimizes $F(\phi)$ directly:

$$\frac{\partial \phi}{\partial t} = \ -\nabla_\phi F(\phi) \quad =: G(\phi). \tag{3}$$

Note that this equation is *a priori* not related to Eq.(2). Since the introduction of the variational level set approach, much work has been carried out under this framework, e.g. [6–8], to name a few. However only very few functions in $L^2(\Omega \to \mathbb{R})$ are SDFs of a shape; for instance, they must satisfy the Eikonal Equation $|\nabla_x \phi| = 1$ almost everywhere. Therefore, a new ϕ obtained from a discrete step of Eq.(3) will generally not itself be a valid SDF. Various authors [5, 9, 10] showed that if the velocity field G in Eq.(3) satisfies

$$\nabla_x G \cdot \nabla_x \phi = 0 \tag{4}$$

then the evolving level set ϕ will always remain a valid SDF for all time. However, a general level set gradient $G = -\nabla_\phi F(\phi)$ is unlikely to satisfy Eq.(4), and several approaches have been proposed to maintain the SDF property of ϕ [11].

The first approach is known as "velocity extension"[9, 10]. A narrow band around Γ is first defined, and Eq.(4) is solved to extend G with the "Fast Marching" method [10]. A second approach is known as "reinitialization" [12]. Here, Eq.(3) is used to update ϕ, but since the newly obtained ϕ will drift away from a SDF, the evolution is occasionally stopped, and the SDF of the zero level set of ϕ is recomputed. A third approach was proposed by Li et al. [13], in which the deviation of $|\nabla_x \phi|$ from 1 is incorporated into the energy function $F(\phi)$.

However, previous works neglect one important issue. Although the desired energy $E(\Gamma)$ can be rewritten as $F(\phi)$, the meaning of the gradient $\nabla_\phi F(\phi)$ is fundamentally different from $\nabla_\Gamma E(\Gamma)$ because the computation $\nabla_\phi F(\phi)$ is performed without the constraint that the gradient should belong to the very particular subset of variations of ϕ that maintain its property of being a SDF. Thus the effect of the gradient $\nabla_\phi F(\phi)$ on the zero level Γ may be completely different from $\nabla_\Gamma E(\Gamma)$ and entirely incorrect. Updating the level set function to maintain the SDF property (e.g. by recomputing it from its zero level) does not change the fact that the newly obtained Γ associated with ϕ is wrong.

In this work, we show that with a simple "velocity projection" step, the level set gradient can be made to exactly match the true gradient of $E(\Gamma)$ with respect to Γ, which we call the "shape gradient". Therefore, with our approach, one can still take the derivative $\nabla_\phi F(\phi)$ of an energy $F(\phi)$ with respect to ϕ, and transform it to obtain the correct shape gradient to deform ϕ. To motivate the discussion, Fig.1 compares our result with the standard variational level set method, where $E(\Gamma)$ is defined as the L^2 distance from ϕ to the SDF ϕ_T corresponding to a target curve Γ_T, i.e. $F(\phi) = ||\phi - \phi_T||^2_{L^2}$. We can see that using the standard (incorrect) level set gradient, the initial curve would shrink to a point, while with our corrected gradient, the initial curve is correctly drawn to Γ_T. For more detailed discussion, please refer to Section 4.
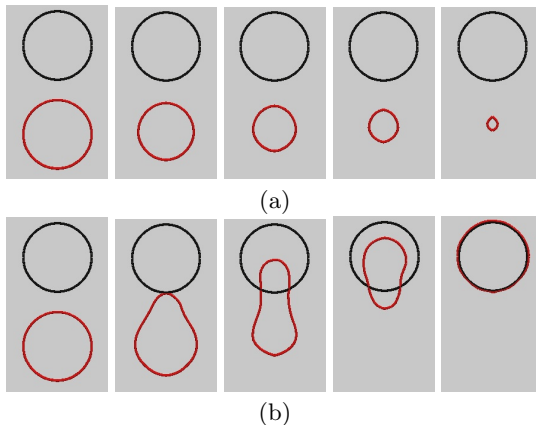

(a)

(b)

Fig. 1: Curve evolution of the red circle Γ, where the black circle Γ_T is the target. The cost function is the L^2 distance between the two SDFs. (a) standard variational level set method, (b) our result.

This paper is organized as follows. In Section 2, we describe the family of admissible SDF variations $\delta\phi$ so that $\phi + \delta\phi$ remains a valid SDF. In Section 3, we use this family to draw the connection between the level set gradient obtained from Eq.(3) and the shape gradient from Eq.(1). We show that by projecting the level set gradient onto the family of admissible SDF variations, we exactly recover the shape gradient. Therefore, with only a simple "velocity projection" step, we can convert any level set gradient to the correct shape gradient and thus deform ϕ by extending the shape gradient with the usual velocity extension approach. We also show how to directly compute the correct deformation of ϕ by integrating the level set gradient over parts of the image Ω, without explicitly computing the shape gradient. In Section 4, we review some often-used level set gradients from the literature that need the "velocity projection" step to make them correct deformations of Γ. We also show experimental results that compare the corrected shape gradient with problematic level set gradients. We then conclude and discuss future directions.

2   The family of admissible SDF variations

Let us consider a closed planar curve $\Gamma \in L^2(\mathbb{S}^1 \to \Omega)$, an infinitesimal deformation field $\delta\Gamma \in L^2(\Gamma \to \mathbb{R}^2)$, which can be seen as a function in $L^2(\mathbb{S}^1 \to \mathbb{R}^2)$ using Γ's parameterization, and let $\phi \in L^2(\Omega \to \mathbb{R})$ be the SDF associated with Γ. If we consider an infinitesimal variation $\delta\phi$ of ϕ, which is any function in $L^2(\Omega \to \mathbb{R})$, in the general case $\phi + \delta\phi$ would not be a valid SDF of some corresponding shape. Thus we should only consider variations $\delta\phi$ so that there exists a shape Γ' so that $\phi + \delta\phi$ is the SDF of it, i.e. $\phi + \delta\phi = \Phi(\Gamma')$.

Let us call $\mathcal{F}$ the family of all such infinitesimal deformations $\delta\phi$. There is a bijection between SDF variations $\delta\phi$ in $\mathcal{F}$ and shape deformations $\delta\Gamma$ of Γ; that is, for any vector field $\delta\Gamma$ normal to Γ at each point of Γ (since tangent displacements do not affect the shape), we can associate a corresponding SDF variation $\delta\phi$ and vice versa. We show in the appendix that to match the shape deformation $\delta\Gamma$, one has to update $\delta\phi$ according to:

$$\forall x \in \Omega, \quad \delta\phi(x) = -\delta\Gamma(s_x) \cdot \boldsymbol{n}_{\Gamma(s_x)} \tag{5}$$

where $\Gamma(s_x)$ is the projection of point x onto Γ, and $\boldsymbol{n}_{\Gamma(s_x)}$ is the unit normal at point $\Gamma(s_x)$ pointing outwards. Here $\delta\phi$ can be understood as $\frac{d\phi}{d\Gamma}(\delta\Gamma)$. Note however that $\frac{d\phi}{d\Gamma}$ is not defined when a topological change occurs, so that ϕ will have to be recomputed from its 0-level after topological changes.

Fig. 2 illustrates the admissible variations of an SDF. Intuitively, Eq.(5) implies that a valid deformation $\delta\phi$ at any point x in Ω depends only on its projection onto Γ: if two points share the same projection point $\Gamma(s_x)$, then their variation will be the same. This is a known result [5, 9, 10]. Consequently, all points on a projection line vary the same way, i.e. $\delta\phi$ is a constant along projection lines to Γ. Conversely, if $\delta\phi$ is constant along all projection lines to Γ, then there exists a deformation $\delta\Gamma$ associated with it. Note that the projection
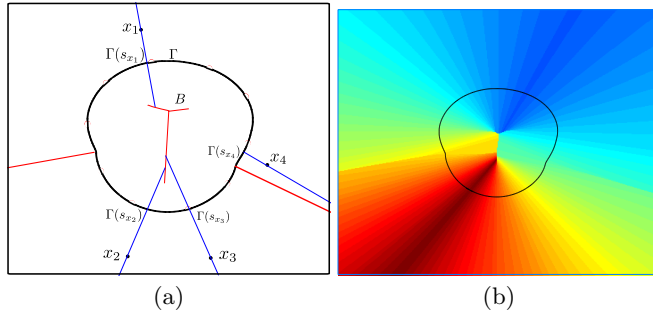
Fig. 2: Illustration of the admissible variations of a SDF. (a) The projection line (in blue) is the line going through x that orthogonally intersects Γ at $\Gamma(s_x)$ and stops at the skeleton B. (b) The admissible variation $\delta\phi$ is a constant along the projection lines.

$\Gamma(s_x)$ is well-defined for all points in Ω except for those on the skeleton of Γ. Since we will later integrate bounded variations over regions of Ω in which the Lebesgue measure of the skeleton is 0, this will pose no problem in our work.

As a consequence, the family $\mathcal{F}$ of all admissible variations of a SDF ϕ is the set of all $L^2(\Omega \to \mathbb{R})$ functions that are constant along projection lines to Γ. This means that, when performing a shape evolution based on a level set representation, one should ensure that the level set variation belongs to this family $\mathcal{F}$. Numerical algorithms such as the Fast Marching method [10] can be used to obtain such a level set variation based on the deformation of Γ.

3   Velocity projection

In the previous section, we defined the family of all admissible variations of an SDF as the set of all functions that are constant along projection lines to Γ. We will now use this result to draw a connection between $\nabla_\phi F(\phi)$, which we call the "level set gradient" and $\nabla_\Gamma E(\Gamma)$, which we call the "shape gradient". We will show that projecting $\nabla_\phi F(\phi)$ onto the family $\mathcal{F}$ will exactly produce $\nabla_\Gamma E(\Gamma)$.

3.1   Gradients and inner products

The gradient definition depends on the choice of the inner product in the tangent space of shapes [14, 15]. In this work, we use the standard L^2 inner product:

$$\langle \delta\Gamma_1 | \delta\Gamma_2 \rangle_{L^2(\mathbb{S}^1 \to \mathbb{R})} \;=\; \int_\Gamma \delta\Gamma_1(s) \cdot \delta\Gamma_2(s)\, d\Gamma(s) \tag{6}$$

where $\delta\Gamma_1$ and $\delta\Gamma_2$ are two deformations of Γ, where s denotes a parameterization of Γ, and where $d\Gamma(s) = \left\| \frac{d\Gamma}{ds} \right\|_{\mathbb{R}^2} ds$ is the associated differential element (i.e. its parameterization-independent length). The gradient associated with this inner product is then defined as the unique deformation $\nabla_\Gamma E(\Gamma)$ that satisfies

$$\forall\, \delta\Gamma, \quad DE(\Gamma)(\delta\Gamma) \;=\; \langle \nabla_\Gamma E(\Gamma) | \delta\Gamma \rangle_{L^2(\mathbb{S}^1 \to \mathbb{R})} \tag{7}$$

where $DE(\Gamma)(\delta\Gamma)$ is the usual directional derivative of E at Γ along the direction $\delta\Gamma$. One of our goals for future work (discussed in the last section) is to extend our current framework to other inner products.

3.2   Relating the two gradients

Since ϕ is a function of Γ (its SDF), and since $E(\Gamma) = F(\phi(\Gamma)) \; \forall \, \Gamma$, we have $DE(\Gamma)(\delta\Gamma) = DF(\phi)(\frac{d\phi}{d\Gamma}(\delta\Gamma))$, i.e. $DE(\Gamma)(\delta\Gamma) = DF(\phi)(\delta\phi)$. On the one side:

$$DE(\Gamma)(\delta\Gamma) \;=\; \langle \nabla_\Gamma E(\Gamma) | \delta\Gamma \rangle_{L^2(\mathbb{S}^1 \to \mathbb{R})}$$

while on the other side, Eq.(5) and the definition of the gradient in Eq.(7) give :

$$DF(\phi)(\delta\phi) = \langle \nabla_\phi F(\phi) | \delta\phi \rangle_{L^2(\Omega \to \mathbb{R})} = \big\langle \nabla_\phi F(\phi) \big| -\delta\Gamma(s_x) \cdot \boldsymbol{n}_{\Gamma(s_x)} \big\rangle_{L^2(\Omega \to \mathbb{R})}$$

so that combining both sides gives:

$$-\int_\Omega \nabla_\phi F(\phi)(x) \; (\delta\Gamma(s_x) \cdot \boldsymbol{n}_\Gamma(s_x)) \; dx \;=\; \int_\Gamma \nabla_\Gamma E(\Gamma)(s) \cdot \delta\Gamma(s) \; d\Gamma(s) \qquad (8)$$

We are now ready to derive a more explicit relation between these two gradients.

As we pointed out, the Lebesgue measure of the skeleton is 0, and under smoothness assumptions about F, the integrand is bounded and consequently the integral over Ω is the same as the integral over $\Omega\backslash\text{Skeleton}(\Gamma)$. We also note that any point $x \in \Omega\backslash\text{Skeleton}(\Gamma)$ not on the skeleton can be written as:

$$x \;=\; \Gamma(s_x) + \phi(x)\boldsymbol{n}_{\Gamma(s_x)} \;=\; \Gamma(s) + r\,\boldsymbol{n}_{\Gamma(s)}$$

where $s(x) = s_x$ and $r(x) = \phi(x)$, which is illustrated in Fig. 3. Note that r can be negative. We will define a new coordinate system using s and r such that the mapping that associates $x \in \Omega\backslash\text{Skeleton}(\Gamma)$ with (s, r) is injective. The infinitesimal (vector) elements of the two coordinate systems are related by:

$$d\mathbf{x} \;=\; \left\|\frac{d\Gamma}{ds}\right\|_{\mathbb{R}^2} \boldsymbol{t}_{\Gamma(s)}\, ds \;+\; \boldsymbol{n}_{\Gamma(s)}\, dr \;-\; r\, \kappa_{\Gamma(s)} \left\|\frac{d\Gamma}{ds}\right\|_{\mathbb{R}^2} \boldsymbol{t}_{\Gamma(s)}\, ds$$

$$=\; (1 - \kappa_{\Gamma(s)}r) \left\|\frac{d\Gamma}{ds}\right\|_{\mathbb{R}^2} \boldsymbol{t}_{\Gamma(s)}\, ds \;+\; \boldsymbol{n}_{\Gamma(s)}\, dr$$

where $\boldsymbol{t}_{\Gamma(s)}$ is the unit tangent of $\Gamma(s)$, and $\kappa_{\Gamma(s)}$ is the curvature of $\Gamma(s)$, which means by definition $\frac{d}{ds}\Gamma(s) = \left\|\frac{d\Gamma}{ds}\right\|_{\mathbb{R}^2} \boldsymbol{t}_{\Gamma(s)}$ and $\frac{d}{ds}\boldsymbol{n}_{\Gamma(s)} = -\kappa_{\Gamma(s)} \left\|\frac{d\Gamma}{ds}\right\|_{\mathbb{R}^2} \boldsymbol{t}_{\Gamma(s)}$. The determinant of the Jacobian $\left|\frac{dx}{ds}, \frac{dx}{dr}\right|$, which is the ratio between the infinitesimal area elements, is then $|1 - \kappa_{\Gamma(s)}r| \left\|\frac{d\Gamma}{ds}\right\|_{\mathbb{R}^2}$. Therefore, the right side of Eq.(8) can be rewritten as:

$$-\int_\Omega \nabla_\phi F(\phi)(x) \; (\delta\Gamma(s_x) \cdot \boldsymbol{n}_{\Gamma(s_x)}) \; dx$$

$$=\; -\int_\Omega \nabla_\phi F(\phi)(x_{(s,r)}) \; (\delta\Gamma(s) \cdot \boldsymbol{n}_{\Gamma(s)}) \, |1 - \kappa_{\Gamma(s)}r| \left\|\frac{d\Gamma}{ds}\right\|_{\mathbb{R}^2} dr\, ds$$

$$=\; -\int_\Gamma \int_{l(s)} \nabla_\phi F(\phi)(x_{(s,r)}) \, |1 - \kappa_{\Gamma(s)}r| \, dr \quad \boldsymbol{n}_{\Gamma(s)} \cdot \delta\Gamma(s) \; d\Gamma(s)$$
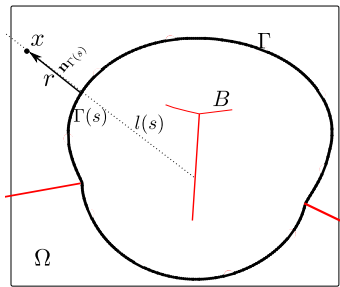
Fig. 3: Illustration of the change of coordinates. See text for explanations.

where $l(s)$ is the projection line that goes through $\Gamma(s)$ as illustrated in Fig. 3. It is the set of all points of Ω whose projection on Γ is $\Gamma(s)$. It is thus a part of a line, which stops at the skeleton of Γ and is also delimited by the boundary of Ω. Therefore, a projection line $l(s)$ is a segment and the integral is well-defined. Since the equality Eq.(8) holds for all possible shape deformations $\delta\Gamma$, we obtain:

$$\nabla_\Gamma E(\Gamma)(s) = -\int_{l(s)} \nabla_\phi F(\phi)(x_{(s,r)}) \left|1 - \kappa_{\Gamma(s)}\, \phi(x_{(s,r)})\right| dr\, \boldsymbol{n}_{\Gamma(s)} \qquad (9)$$

with $\phi(x_{(s,r)})$ being just r by definition. This is the key contribution of our work, since it draws the connection between the shape gradient $\nabla_\Gamma E(\Gamma)$ and the level set gradient $\nabla_\phi F(\phi)$ frequently used in the literature. The intuitive explanation of Eq.(9) is that the shape gradient $\nabla_\Gamma E(\Gamma)$ at $\Gamma(s)$ is a weighted integral of the level set gradient $\nabla_\phi F(\phi)$ along the projection line going through $\Gamma(s)$. We will shortly introduce a natural interpretation of these weights.

3.3   The correct way to evolve the level sets

Eq.(9) shows how to calculate the shape gradient $\nabla_\Gamma E(\Gamma)$ from the level set gradient $\nabla_\phi F(\phi)$. However, to actually update the level sets, we need to find the corresponding variation of ϕ, that is $\delta\phi$. One possibility for this is to use the classical "velocity extension" approach [9, 10] where the velocity defined on Γ is extended to Ω. This involves the computation of the zero level set, which is sometimes undesirable. Another way is to directly express $\delta\phi$ using Eq.(5):

$$\delta\phi(x) = -\delta\Gamma(s_x) \cdot \boldsymbol{n}_{\Gamma(s_x)}$$
$$= -\int_{l(s_x)} \left|1 - \kappa(s_x)\, \phi(x'_{(s_x,r)})\right| \nabla_\phi F(\phi)(x'_{(s_x,r)})\, dr \qquad (10)$$

To compute the variation $\delta\phi(x)$ at point x, it is thus sufficient to integrate the level set gradient, weighted by the area element ratio, along the projection line $l(s_x)$ that shares the same projection point as x. We will now give a natural geometrical interpretation of Eq.(9) and Eq.(10) which will lead to our numerical implementation algorithm.

3.4   Implementation

Let us first examine the term $|1 - r\ \kappa_{\Gamma(s)}|$ in Eq.(9). From basic differential geometry, we have that $\kappa_{\Gamma(s)} = \frac{d\theta}{ds}\left\|\frac{d\Gamma(s)}{ds}\right\|^{-1} = \frac{1}{R}$, where R is the radius of the osculating circle at $\Gamma(s)$, with the same sign as the curvature, and where $d\theta$ is the angle formed by the normals to the curve at $\Gamma(s)$ and $\Gamma(s+ds)$, as illustrated in Fig. 4a. We also note that $r = \phi(x_{(s,r)})$ and is negative when x is inside Γ, and positive when x is outside Γ. We can show that:

$$|1 - \phi(x)\kappa_{\Gamma(s)}|\ d\Gamma(s) = \left|1 - \frac{r}{R}\right| d\Gamma(s) = \frac{|r - R|}{R}\ d\Gamma(s) = |r - R| d\theta = dL \quad (11)$$

where dL is the distance $x_{(s,r)}$ will travel for an infinitesimal step ds, i.e. the length of the arc formed by the projection lines through $\Gamma(s)$ and $\Gamma(s+ds)$, at a distance $|r - R|$ from their intersection, as shown in Fig. 4a. Note that $R < 0$ in this figure. Since in Eq.(9) we are integrating a function of x times $dL(r)$ along the projection line segment bounded on one side by the skeleton, we are interested in the region formed by the skeleton, the boundary of Ω and the projection lines going through $\Gamma(s)$ and $\Gamma(s+ds)$, that is, the red dotted region dW shown in Fig. 4b. One can indeed show that, for any smooth function f:

$$\int_{dW(ds)} f(x)\ dx = \int_{r \in l(s)} f(x_{(s,r)})\ dL(r)\ dr\ +\ o\left(\|\nabla_x f\|_\infty ds\right)$$
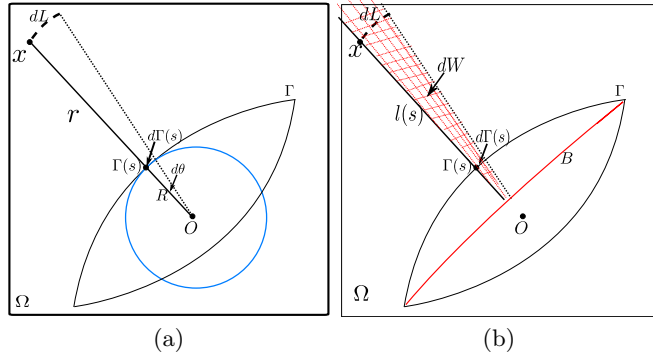

(a)   (b)

Fig. 4: Another look at velocity projection. (a) The osculating circle of $\Gamma(s)$ is shown in blue, where O is the center of the osculating circle. (b) The skeleton of Γ is B, shown in red. The area of the dashed subregion dW is $\int_{r \in l_s} dL(r)\ dr = \int_{r \in l_s} |1 - r\ \kappa_{\Gamma(s)}| dr\ d\Gamma(s)$.

The above analysis shows that Eq.(9) can be written as an integral over this subregion dW when multiplied by an infinitesimal step $d\Gamma(s)$. We believe this is a more intuitive explanation of velocity projection, i.e. that the shape gradient

at any point $\Gamma(s)$ is the average limit of the level set gradient over the region whose projection points are between $\Gamma(s)$ and $\Gamma(s + ds)$:

$$\nabla_\Gamma E(\Gamma)(s) \;\; = \;\; -\lim_{ds \to 0} \; \frac{1}{d\Gamma(s)} \; \int_{dW(ds)} \nabla_\phi F(\phi)(x) \; dx \;\; \boldsymbol{n}_{\Gamma(s)} \qquad (12)$$

Although both Eq.(5) and Eq.(12) are the same theoretically, we recommend using the integral over dW for the following reason. To integrate along $l(s)$, we need to find the explicit range of r, whose estimation is not straightforward since it depends on the skeleton. We can however avoid the estimation of the skeleton if we integrate over regions W.

For any point x in Ω, we can easily locate its projection point $\Gamma(s_x) = x - \phi(x) \nabla_x \phi(x)$. Suppose that Γ is discretized by points $\Gamma(s_i), i = 1, \ldots, N$, and there is a W_i associated with each $\Gamma(s_i)$. Then Eq.(12) could be discretized as:

$$\nabla_\Gamma E(\Gamma)(s_i) = - \sum_{x \in W_i} \nabla_\phi F(\phi)(x) \;\; \boldsymbol{n}_{\Gamma(s_i)} \qquad (13)$$

with its equivalent from Eq.(10) for direct level set evolutions, if $x \in W_i$:

$$\delta\phi(x) \;\; = \;\; \sum_{y \in W_i} \nabla_\phi F(\phi)(y) \qquad (14)$$

which means that the correct level set evolution can be computed from the level set gradient $\nabla_\phi F(\phi)$ very easily by just integrating over regions that share similar projection points.

However, since the projection s_x of a random point $x \in \Omega$ is unlikely to be exactly one of the s_i, a point x will typically contribute to more than one W_i. Let us denote by h_i^x the weight that x contributes to W_i. We have the constraint that all contributions of a point sum up to $1 : \sum_i h_i^x = 1$. Thus Eq.(13) is replaced by:

$$\nabla_\Gamma E(\Gamma)(s_i) = - \sum_{x \in \Omega} h_i^x \, \nabla_\phi F(\phi)(x) \;\; \boldsymbol{n}_{\Gamma(s_i)} \qquad (15)$$

or, more practically, we obtain the level set evolution from Eq.(10):

$$\delta\phi(x) \;\; = \;\; \sum_i h_i^x \sum_{y \in \Omega} h_i^y \, \nabla_\phi F(\phi)(y). \qquad (16)$$

Then the problem comes down to how to estimate the weight h_i^x that x contributes to W_i in a computationally effective manner. In practice, we assign h_i^x to a function of the distance from $\Gamma(s_x)$ to $\Gamma(s_i)$ and normalize accordingly. A better numerical implementation algorithm is also one of our future goals.

4   Implications for common level set gradients

As mentioned earlier, much work has been carried out under the variational level set method, without considering whether the level set gradient agrees with the

shape gradient. In this section, we will discuss some energy models that depend on SDFs and their gradients. We will not discuss energy models that depend only on Γ, such as the Geodesic Active Contour [16] and the Chan-Vese models [6], since our aim is to compare shape gradients with level set gradients.

We first consider the L^2 distance between two SDFs and its gradient:

$$F(\phi) = ||\phi - \phi_T||^2_{L^2}, \quad \nabla_\phi F(\phi) = 2(\phi - \phi_T) \tag{17}$$

Here ϕ_T is a target SDF. This energy is important and has many applications in shape analysis, morphing and shape prior image segmentation [17, 18]. We are not aware of any work on computing the corresponding $E(\Gamma)$ or its shape gradient $\nabla_\Gamma E(\Gamma)$. Charpiat et al. [19] showed how to calculate the shape gradient directly with the $W^{1,2}$ norm but under a smooth approximation of infima. We can easily compute the shape gradient with our velocity projection step.

If instead, as in other works, we let ϕ evolve with the level set gradient, and rebuild it regularly from its 0-level to maintain its SDF property, we notice the following effect. Curve segments of Γ that lie inside the region delimited by Γ_T expand, while segments of Γ that lie outside of Γ_T shrink. This immediately implies that if Γ lies completely outside of Γ_T, then the evolution process will shrink Γ until it disappears no matter how close they are. This phenomenon is illustrated in Fig. 1a where we are trying to evolve the red circle Γ to the black circle Γ_T. With the velocity projection approach, we can calculate the shape gradient and deform Γ accordingly. Fig. 1b illustrates the deformation process under the correct shape gradient, which naturally morphs the initial curve to Γ_T.
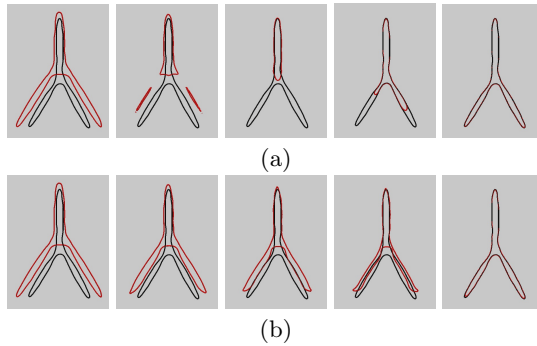

Fig. 5: Curve evolution of the red curve Γ, where the black curve Γ_T is the target. The cost function is the L^2 distance between the two SDFs. (a) standard variational level set method, (b) our result.

Fig. 5 illustrates the evolution of two overlapped shapes using the same L^2 distance with and without velocity projection. Fig. 5a illustrates the traditional evolution without velocity projection. As we can see, the parts that are outside Γ_T will shrink while the parts that are inside will expand. Fig. 5b illustrates the

correct deformation with velocity projection. As we can see, the deformation is more meaningful, leading to much better point correspondences. However, this model suffers from the drawback that the energy and thus the gradient depend on the domain Ω. That is, by fixing Γ and Γ_T and changing Ω alone, we will get different energies, gradients and thus different deformation processes.

As a second example, consider the following energy model:

$$F(\phi) = \int_{\Omega} (\phi - \phi_T)^2 H(-\phi) dx \tag{18}$$

which is the integration of the L^2 distance between ϕ and ϕ_T inside Γ. Here H is the Heaviside function. This energy model was studied by Rousson and Paragios [7]. It can be shown that the level set gradient is:

$$\nabla_\phi F(\phi) = 2(\phi - \phi_T)H(-\phi) - (\phi - \phi_T)^2 \delta(\phi) \tag{19}$$

In this case, the velocity projection step is necessary to calculate the correct shape gradient. The evolution process of the correct shape gradient is illustrated in Fig. 6a. Since the integration is only inside Γ, it is not appropriate if Γ lies outside Γ_T. To improve the evolution, we study the following symmetric term:

$$F(\phi) = \int_{\Omega} (\phi - \phi_T)^2 H(-\phi_T) dx \tag{20}$$

which is the integration of the L^2 distance between ϕ and ϕ_T inside Γ_T. This energy was studied by Cremers and Soatto [8]. The level set gradient is:

$$\nabla_\phi F(\phi) = 2(\phi - \phi_T)H(-\phi_T) \tag{21}$$

It is only defined within Γ_T and therefore if we try to make Γ evolve to Γ_T under this gradient alone, most likely it won't move at all! We can again calculate its shape gradient (see Fig. 6b). This evolution does not correctly draw Γ to Γ_T. The reason is that the level set gradient is non-0 inside Γ_T and all the projection points inside Γ_T fall only on the blue curve segment of Γ in Fig. 6c. Therefore, the fastest way to minimize this energy, or the gradient, is to only move the blue curve segment to Γ_T. The dissimilarity measure between Γ and Γ_T can be made symmetric by combining both Eq.(18) and Eq.(20):

$$F(\phi) = \int_{\Omega} (\phi - \phi_T)^2 H(-\phi_T) dx + \int_{\Omega} (\phi - \phi_T)^2 H(-\phi) dx \tag{22}$$

The evolution of this symmetric dissimilarity measure is shown in Fig. 6d. As we can see, it correctly evolves Γ to Γ_T. We note here that the traditional variational gradient of this symmetric measure could also be used to evolve Γ. However, since the level set gradient Eq.(21) is not defined on curve segments of Γ that lie outside Γ_T, only the gradient Eq.(19) would play a role in the evolution and it would not correctly draw Γ to Γ_T.
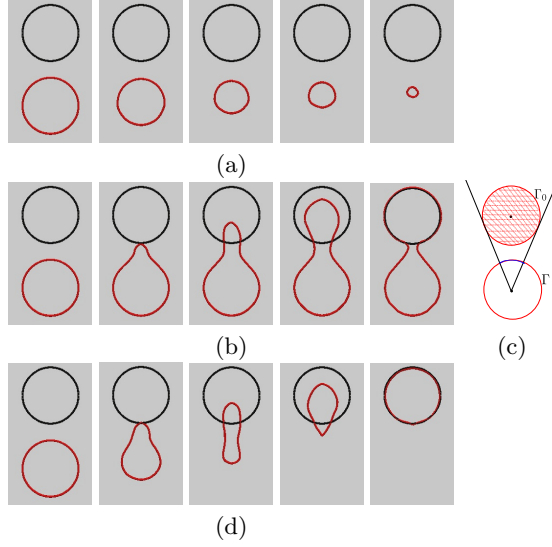
Fig. 6: The evolution from the red circle Γ to the black circle Γ_T under the correct shape gradient of the three different energy models. (a) energy model Eq.(18), (b) energy model Eq.(20), (c) the blue curve segment is the region that $H(-\phi_T)$ projects onto for (b), (d) energy model Eq.(22).

As a final example, we consider the following energy model [20]:

$$E(\Gamma) = \int_{\Gamma_T} \phi^2 \, ds = \int_{\Omega} \phi^2 \, \delta(\phi_T) \, dx = F(\phi) \qquad (23)$$

This also defines a dissimilarity between Γ and Γ_T . The level set gradient is:

$$\nabla_\phi F(\phi) = 2\phi \, \delta(\phi_T) \qquad (24)$$

which is defined only along Γ_T and is zero everywhere else. Therefore, it is also problematic for the traditional variational level set method. However, if we apply the velocity projection approach we can calculate the true deformation field:

$$\delta\phi(x) = -2 \sum_{y \in (l(s_x) \cap \Gamma_T)} |1 - \phi(y)\kappa(s_x)| \, \phi(y) \qquad (25)$$

where $l(s_x)$ is the projection line going through x. The deformation process is illustrated in Fig. 7a. However, as we can see, this evolution also does not correctly draw Γ to Γ_T. The reason is the same as in Fig. 6c.

To improve the evolution process, we can add a second symmetric term:

$$F(\phi) = \int_{\Gamma_T} \phi^2 \, ds + \int_{\Gamma} \phi_T^2 \, ds \qquad (26)$$

The evolution of this symmetric energy under the correct shape gradient is illustrated in Fig. 7c and it successfully draws Γ to Γ_T.
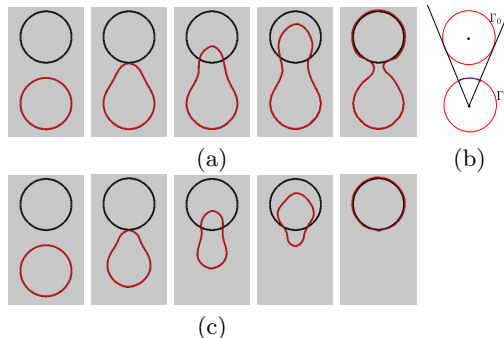
Fig. 7: The evolution from the red circle Γ to the black circle Γ_T under the shape gradient of two different energy models. (a) energy model Eq.(23), (b) the blue curve segment is the region that $\delta(\phi_T)$ projects onto, (c) energy model Eq.(26).

5 Discussion and conclusions

The experiments in Section 4 show that the limitations of traditional variational level set formulations can be fixed with our velocity projection step. In this work, we used shape morphing as a motivating application since it is closely related to other computer vision problems. In future work, we plan to apply our method to shape-prior image segmentation and statistical shape analysis. Fig. 5 shows the evolution of tubular structures under the traditional level set gradient can be problematic and our corrected shape gradient can handle these cases nicely.

We should point out that the geometric $L^2(\mathbb{S}^1 \to \mathbb{R})$ inner product has been shown to suffer from serious drawbacks as a metric on the manifold of shapes [21, 22]. Specifically, the L^2 geodesic distance between two shapes is 0. Since we consider gradient descents only, the L^2 inner product will pose no theoretical problem. However, we would like to investigate other inner products such as the $H^1(\mathbb{S}^1 \to \mathbb{R})$ inner product [14, 15]. We are also investigating energy models other than $F(\phi)$, such as $F(\Gamma, \phi)$, as well as extending our framework to 3D.

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Appendix

Proof of Eq.(5): $(\phi + \delta\phi)(x) = \phi_{\Gamma'}(x)$ implies, using the signed distance d:

$$\delta\phi(x) = \phi_{\Gamma'}(x) - \phi(x) = d(x, \Gamma') - d(x, \Gamma). \tag{27}$$

Since the part of an infinitesimal deformation that is tangent to Γ has no effect on the shape (just reparameterizes), we only keep the part of the deformation that is normal to Γ, and the following redefinition of Γ' describes the same shape (as a set of points): $\Gamma'(s) := \Gamma(s) + (\delta\Gamma(s) \cdot \mathbf{n}_{\Gamma(s)})\mathbf{n}_{\Gamma(s_x)}$.

Then $x - \Gamma'(s_x) = x - \Gamma(s_x) - (\delta\Gamma(s_x) \cdot \mathbf{n}_{\Gamma(s_x)})\mathbf{n}_{\Gamma(s_x)}$

$(x - \Gamma'(s_x)) \cdot \mathbf{n}_{\Gamma(s_x)} = (x - \Gamma(s_x)) \cdot \mathbf{n}_{\Gamma(s_x)} - (\delta\Gamma(s_x) \cdot \mathbf{n}_{\Gamma(s_x)})(\mathbf{n}_{\Gamma(s_x)} \cdot \mathbf{n}_{\Gamma(s_x)})$

$d(x, \Gamma') = d(x, \Gamma) - \delta\Gamma(s_x) \cdot \mathbf{n}_{\Gamma(s_x)}$

since the projection of any point $x \in \Omega$ on the closed subset Γ is necessarily orthogonal to its boundary Γ. Hence with Eq.(27) one has $\delta\phi(x) = -\delta\Gamma(s_x) \cdot \mathbf{n}_{\Gamma(s_x)}$.