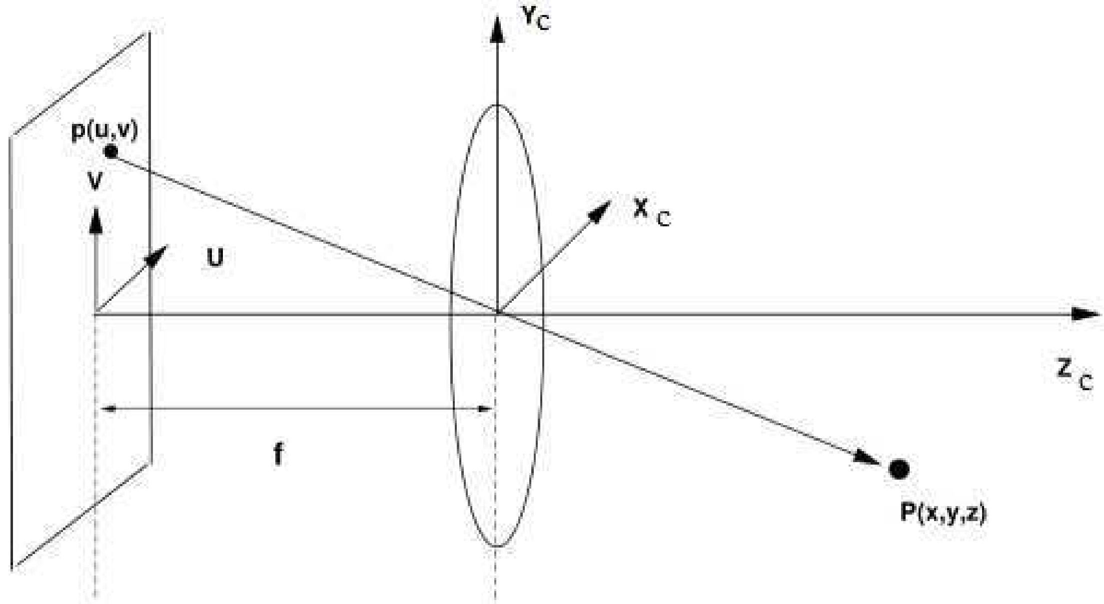
Camera Projection Models

We will introduce different camera projection models that relate the location of an image point to the coordinates of the corresponding 3D points. The projection models include: full perspective projection model, weak perspective projection model, affine projection model, and orthographic projection model.

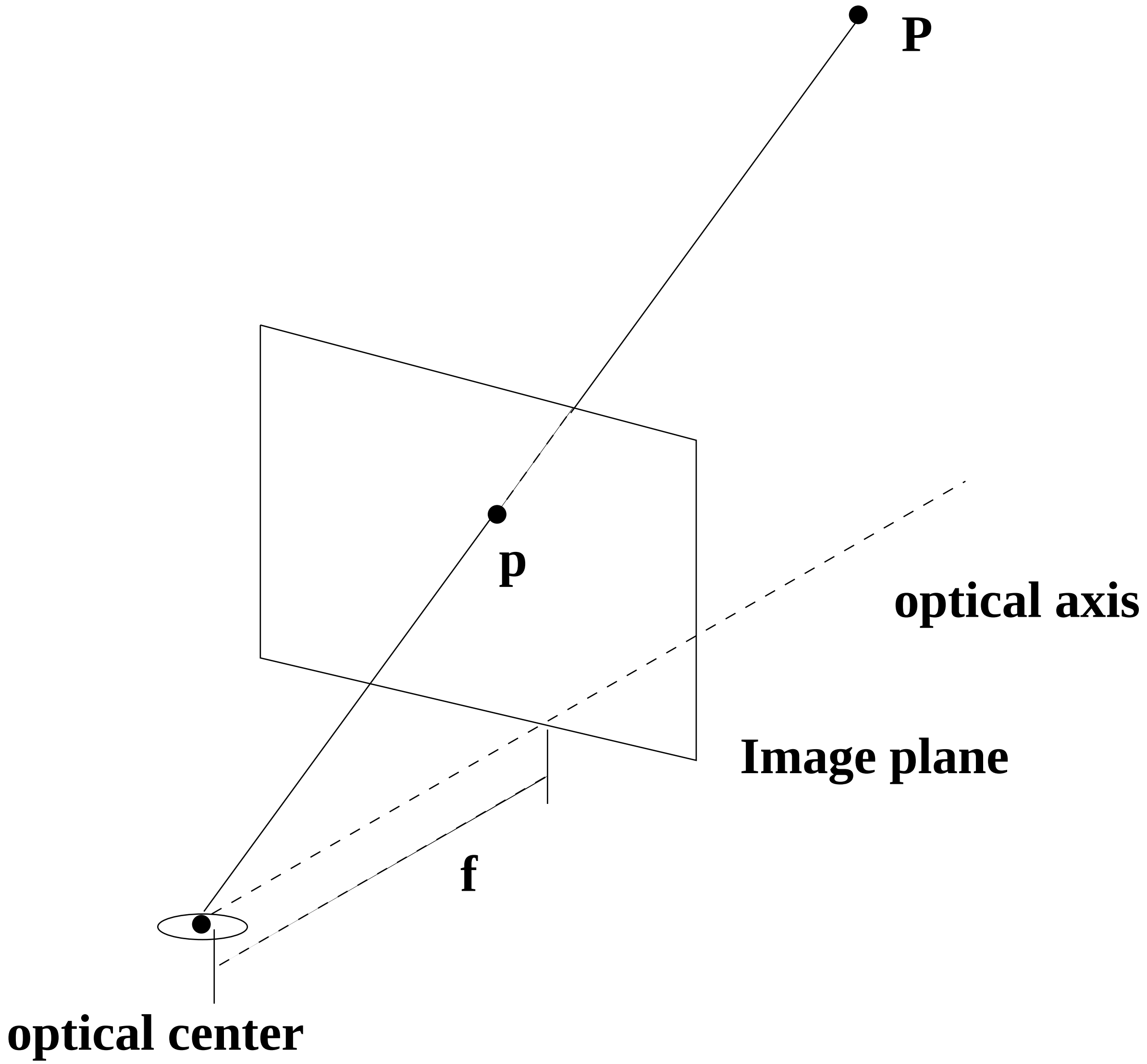
The Pinhole Camera Model



Based on simply trigonometry (or using 3D line equations), we can derive

$$u = \frac{-f x_c}{z_c} \qquad v = \frac{-f y_c}{z_c}$$

The Computer Vision Camera Model



$$u = \frac{f x_c}{z_c} \qquad v = \frac{f y_c}{z_c}$$

where $\frac{f}{z_c}$ is referred to as isotropic scaling.The full perspective projection is non-linear.

Weak Perspective Projection

If the relative distance δz_c (scene depth) between two points of a 3D object along the optical axis is much smaller than the average distance $\bar{z}_c$ to the camera ($\delta z < \frac{\bar{z}}{20}$), i.e, $z_c \approx \bar{z}_c$

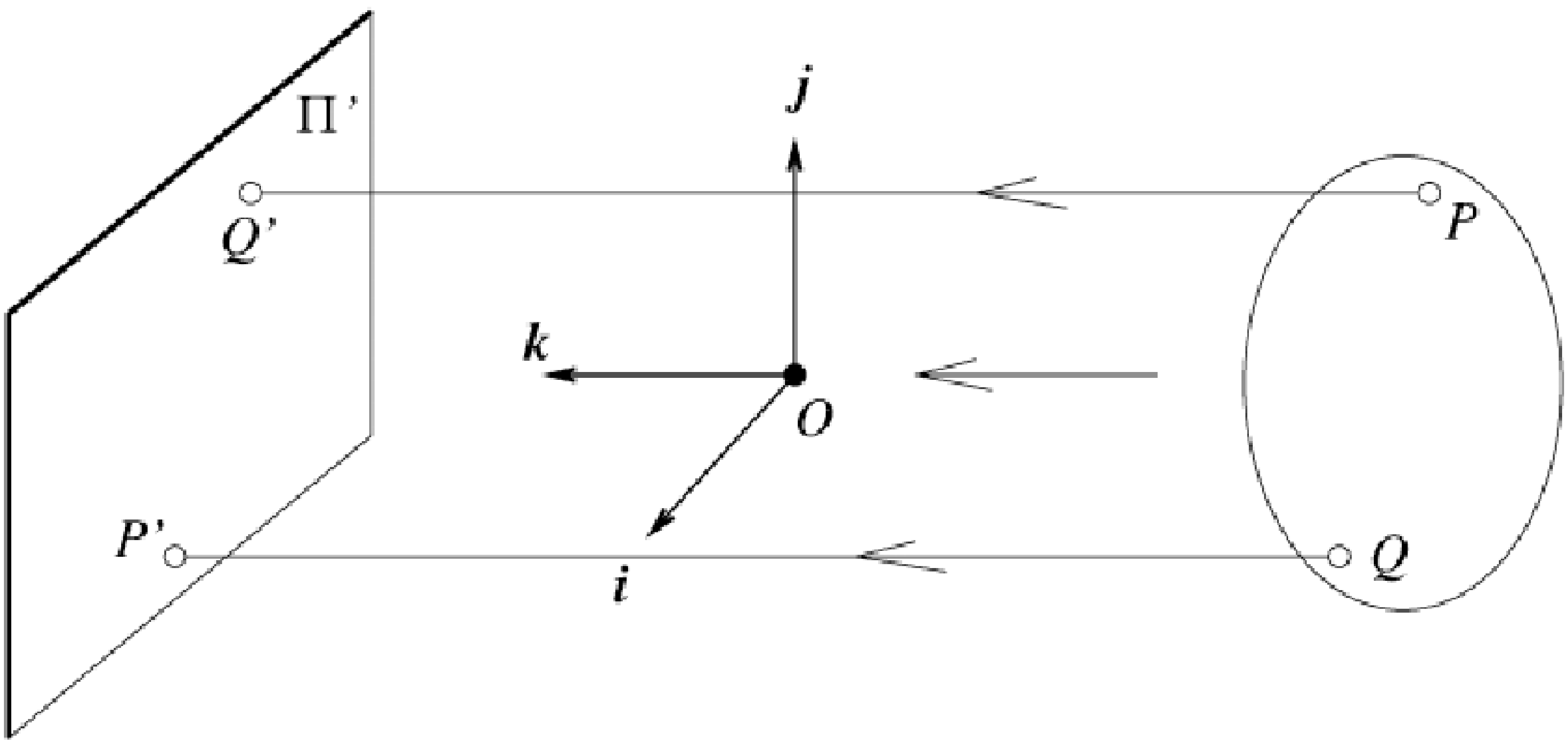
then

$$\begin{aligned} u &= f\frac{x_c}{z_c} \approx \frac{f x_c}{\bar{z}_c} \\ v &= f\frac{y_c}{z_c} \approx \frac{f y_c}{\bar{z}_c} \end{aligned}$$

We have linear equations since all projections have the same scaling factor.

Orthographic Projection

As a special case of the weak perspective projection, when $\frac{f}{z_c}$ factor equals 1, we have $u = x_c$ and $v = y_c$, i.e., the lins (rays) of projection are parallel to the optical axis, i.e., the projection rays meet in the infinite instead of lens center. This leads to the sizes of image and the object are the same. This is called orthgraphic projection.


Π'
j
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Perspective projection geometry

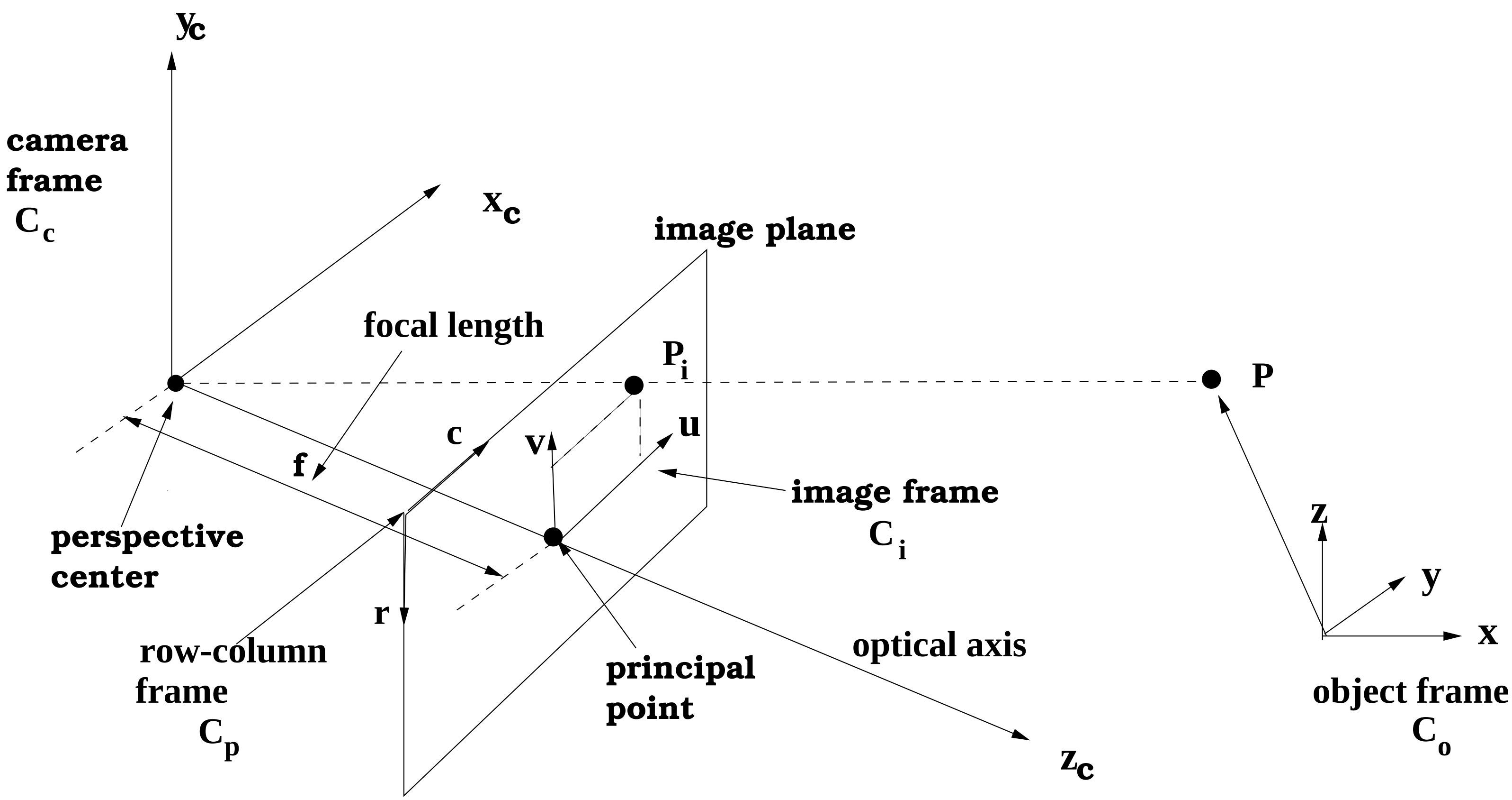


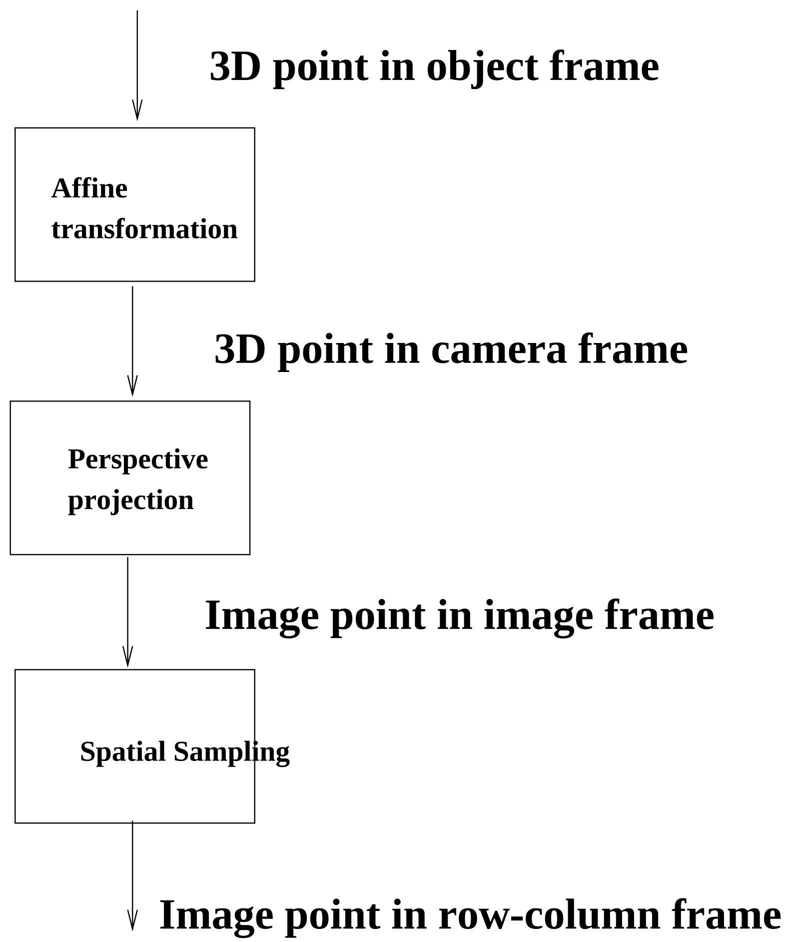
Figure 1: Perspective projection geometry

Notations

Let $P = (x\ y\ z)^t$ be a 3D point in object frame and $U = (u\ v)^t$ the corresponding image point in the image frame before digitization. Let $X_c = (x_c\ y_c\ z_c)^t$ be the coordinates of P in the camera frame and $p = (c\ r)^t$ be the coordinates of U in the row-column frame after digitization.

Projection Process

Our goal is to go through the projection process to understand how an image point (c, r) is generated from the 3D point (x, y, z).



Relationships between different frames

Between camera frame (C_c) and object frame (C_o)

$$X_c = RX + T \tag{1}$$

X is the 3D coordinates of P w.r.t the object frame. R is the rotation matrix and T is the translation vector. R and T specify the orientation and position of the object frame relative to the camera frame.

R and T can be parameterized as

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \qquad T = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}$$

$r_i = (r_{i1}, r_{i2}, r_{i3})$ be a 1 x 3 row vector, R can be written as

$$R = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

Substituting the parameterized T and R into equation 1 yields

$$\begin{pmatrix} x_c \\ y_x \\ z_c \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix} \quad (2)$$

- Between image frame (C_i) and camera frame (C_c)

  Perspective Projection:

$$\begin{aligned} u &= \frac{f x_c}{z_c} \\ v &= \frac{f y_c}{z_c} \end{aligned}$$

Hence,

$$X_c = \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \\ f \end{pmatrix} \tag{3}$$

where $\lambda = \frac{z_c}{f}$ is a scalar and f is the camera focal length.

Relationships between different frames (cont'd)

- Between image frame (C_i) and row-col frame (C_p) (spatial quantization process)

$$\begin{pmatrix} c \\ r \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} c_0 \\ r_0 \end{pmatrix} \tag{4}$$

where s_x and s_y are scale factors ($pixels/mm$) due to spatial quantization. c_0 and r_0 are the coordinates of the principal point in pixels relative to C_p

Collinearity Equations

Combining equations 1 to 4 yields

$$c = s_x f \frac{r_{11}x + r_{12}y + r_{13}z + t_x}{r_{31}x + r_{32}y + r_{33}z + t_z} + c_0$$

$$r = s_y f \frac{r_{21}x + r_{22}y + r_{23}z + t_y}{r_{31}x + r_{32}y + r_{33}z + t_z} + r_0$$

Homogeneous system: perspective projection

In homogeneous coordinate system, equation 3 may be rewritten as

$$\lambda \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} \qquad (5)$$

Note $\lambda = z_c$.

Homogeneous System: Spatial Quantization

Similarly, in homogeneous system, equation 4 may be rewritten as

$$\begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & c_0 \\ 0 & s_y & r_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \tag{6}$$

Homogeneous system: quantization + projection

Substituting equation 5 into equation 6 yields

$$\lambda \begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = \begin{pmatrix} s_x f & 0 & c_0 & 0 \\ 0 & s_y f & r_0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix} \tag{7}$$

where $\lambda = z_c$.

Homogeneous system: Affine Transformation

In homogeneous coordinate system, equation 2 can be expressed as

$$\begin{pmatrix} x_c \\ y_c \\ z_c \\ 1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & r_{13} & t_x \\ r_{21} & r_{22} & r_{23} & t_y \\ r_{31} & r_{32} & r_{33} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \tag{8}$$

Homogeneous system: full perspective

Combining equation 8 with equation 7 yields

$$\lambda \begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} s_x f r_1 + c_0 r_3 & s_x f t_x + c_0 t_z \\ s_y f r_2 + r_0 r_3 & s_y f t_y + r_0 t_z \\ r_3 & t_z \end{pmatrix}}_{P} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \qquad (9)$$

where r_1, r_2, and r_3 are the row vectors of the rotation matrix R, $\lambda = z_c$ is a scaler and matrix P is called the homogeneous projection matrix.

$$P = WM$$

where

$$W = \begin{pmatrix} fs_x & 0 & c_0 \\ 0 & fs_y & r_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} R & T \end{pmatrix}$$

W is often referred to as the intrinsic matrix and M as exterior matrix.

Since $P = WM = [WR \;\; WT]$, for P to be a projection matrix, $Det(WR) \neq 0$, i.e., $Det(W) \neq 0$.

Weak Perspective Camera Model

For weak perspective projection, we have $z_c \approx \bar{z}_c$, i.e., $\bar{z}_c \approx r_3^t \bar{X} + t_z$ Hence,

$$
\begin{aligned}
u &= \frac{f x_c}{\bar{z}_c} \\
v &= \frac{f y_c}{\bar{z}_c}
\end{aligned}
$$

Hence,

$$
\begin{pmatrix} u \\ v \end{pmatrix} = \frac{f}{\bar{z}_c} \begin{pmatrix} x_c \\ y_c \end{pmatrix}
$$

Or

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \frac{f}{\bar{z}_c} \begin{pmatrix} x_c \\ y_c \\ \frac{\bar{z}_c}{f} \end{pmatrix}$$

Since

$$\begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & c_0 \\ 0 & s_y & r_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

We have

$$\begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = \frac{f}{\bar{z}_c} \begin{pmatrix} s_x & 0 & c_0 \\ 0 & s_y & r_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_c \\ y_c \\ \frac{\bar{z}_c}{f} \end{pmatrix}$$

Since

$$\begin{pmatrix} x_c \\ y_c \end{pmatrix} = [R_2 \;\; T_2] \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

where R_2 is the first two rows of R and T_2 is the first two elements of T. Or

$$\begin{pmatrix} x_c \\ y_c \\ \frac{\bar{z}_c}{f} \end{pmatrix} = \begin{pmatrix} R_2 & & & T_2 \\ 0 & 0 & 0 & \frac{\bar{z}_c}{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = \frac{f}{\bar{z}_c} \begin{pmatrix} s_x & 0 & c_0 \\ 0 & s_y & r_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_2 & & & T_2 \\ 0 & 0 & 0 & \frac{\bar{z}_c}{f} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

$$
= \frac{1}{\bar{z}_c} \begin{pmatrix} s_x & 0 & c_0 \\ 0 & s_y & r_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} fR_2 & & & fT_2 \\ 0 & 0 & 0 & \bar{z}_c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
$$

$$
= \frac{1}{\bar{z}_c} \begin{pmatrix} fs_x & 0 & c_0 \\ 0 & fs_y & r_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_2 & & & T_2 \\ 0 & 0 & 0 & \bar{z}_c \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}
$$

Weak Perspective Camera Model

The weak perspective projection matrix is

$$P_{weak} = \begin{pmatrix} f s_x r_1 & f s_x t_x + c_0 \bar{z}_c \\ f s_y r_2 & f s_y t_y + r_0 \bar{z}_c \\ 0^{1\times 3} & \bar{z}_c \end{pmatrix} \quad (10)$$

where r_1 and r_2 are the first two rows of R_2 and $\bar{z}_c = r_3 \bar{X} + t_z$.

Weak Projection Camera Model

Another possible solution is as follows

given $z_c = \bar{z}_c = r_3^t \bar{x} + t_z \quad \bar{x} = (x\ y\ z)^t$

$$\lambda \begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = W \begin{bmatrix} R & T \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = W \begin{bmatrix} r_1 & t_x \\ r_2 & t_y \\ r_3 & t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = W \begin{bmatrix} r_1 & t_x \\ r_2 & t_y \\ 0 & r_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix} + t_z \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= W \begin{bmatrix} r_1 & t_x \\ r_2 & t_y \\ 0 & z_c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = W \begin{bmatrix} r_1 & t_x \\ r_2 & t_y \\ 0 & \bar{z}_c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P_{weak} = W \begin{bmatrix} r_1 & t_x \\ r_2 & t_y \\ 0 & \bar{z}_c \end{bmatrix} = \begin{bmatrix} fs_x & 0 & c_0 \\ 0 & fs_y & r_0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_1 & t_x \\ r_2 & t_y \\ 0 & \bar{z}_c \end{bmatrix}$$

$$= \begin{bmatrix} fs_x r_1 & fs_x t_x + c_0 \bar{z}_c \\ fs_y r_2 & fs_y t_y + r_0 \bar{z}_c \\ 0 & \bar{z}_c \end{bmatrix}$$

Affine Camera Model

A further simplification from weak perspective camera model is the affine camera model, which is often assumed by computer vision researchers due to its simplicity. The affine camera model assumes that the object frame is located on the centroid of the object being observed. As a result, we have $\bar{z}_c \approx t_z$, the affine perspective projection matrix is

$$P_{affine} = \begin{pmatrix} s_x f r_1 & s_x f t_x + c_0 t_z \\ s_y f r_2 & s_y f t_y + r_0 t_z \\ 0 & t_z \end{pmatrix} \tag{11}$$

Affine camera model represents the first order approximation of the full perspective projection camera model. It still only gives an approximation and is no longer useful when the object is close to the camera or the camera has a wide angle of view.

Orthographic Projection Camera Model

Under orthographic projection, projection is parallel to the camera optical axis.

therefore we have

$$\begin{aligned} u &= x_c \\ v &= y_c \end{aligned}$$

which can be approxmiated by $\frac{f}{z_c} \approx 1$.

The orthographic projection matrix can therefore be obtained as

$$P_{orth} = \begin{pmatrix} s_x r_1 & s_x t_x + c_0 \\ s_y r_2 & s_y t_y + r_0 \\ 0 & 1 \end{pmatrix} \qquad (12)$$

Non-full perspective Projection Camera Model

The weak perspective projection, affine, and orthographic camera model can be collectively classified as *non-perspective projection* camera model. In general, the projection matrix for the non-perspective projection camera model

$$\lambda \begin{pmatrix} c \\ r \\ 1 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 0 & p_{34} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

Dividing both sides by p_{34} (note $\lambda = p_{34}$) yields

$$\begin{pmatrix} c \\ r \end{pmatrix} = M_{2\times 3} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

where $m_{ij} = p_{ij}/p_{34}$ and $v_x = p_{14}/p_{34}$, $v_y = p_{24}/p_{34}$

For any given reference point (c_r, r_r) in image and (x_0, y_0, z_0) in space, the relative coordinates $(\bar{c}, \bar{r})$ in image and $(\bar{x}, \bar{y}, \bar{z})$ in space are

$$\begin{pmatrix} \bar{c} \\ \bar{r} \end{pmatrix} = \begin{pmatrix} c - c_r \\ r - r_r \end{pmatrix} \text{ and } \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} x - x_r \\ y - y_r \\ z - z_r \end{pmatrix}$$

It follows that the basic projection equation for the affine and weak perspective model in terms of relative coordinates is

$$\begin{pmatrix} \bar{c} \\ \bar{r} \end{pmatrix} = M_{2\times 3} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}$$

An non-perspective projection camera $M_{2\times 3}$ has 3 independent parameters. The reference point is often chosen as the centroid since centroid is preserved under either affine or weak perspective projection.

Given the weak projection matrix P,

$$P = \begin{pmatrix} fs_x r_1 & fs_x t_x + c_0 \bar{z}_c \\ fs_y r_2 & fs_y t_y + r_0 \bar{z}_c \\ 0 & \bar{z}_c \end{pmatrix}$$

The M matrix is

$$M = \begin{pmatrix} \frac{fs_x r_1}{\bar{z}_c} \\ \frac{fs_y r_2}{\bar{z}_c} \end{pmatrix}$$

$$= \frac{f}{\bar{z}_c} \begin{pmatrix} s_x r_1 \\ s_y r_2 \end{pmatrix}$$

$$= \frac{f}{\bar{z}_c} \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

For affine projection, $\bar{z}_c = t_z$, for orthographic projection, $\frac{f}{\bar{z}_c} = 1$. If we assume $s_x = s_y$, then

$$M = \frac{f s_x}{\bar{z}_c} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

Then, we have only four parameters: three rotation angles and a scale factor.

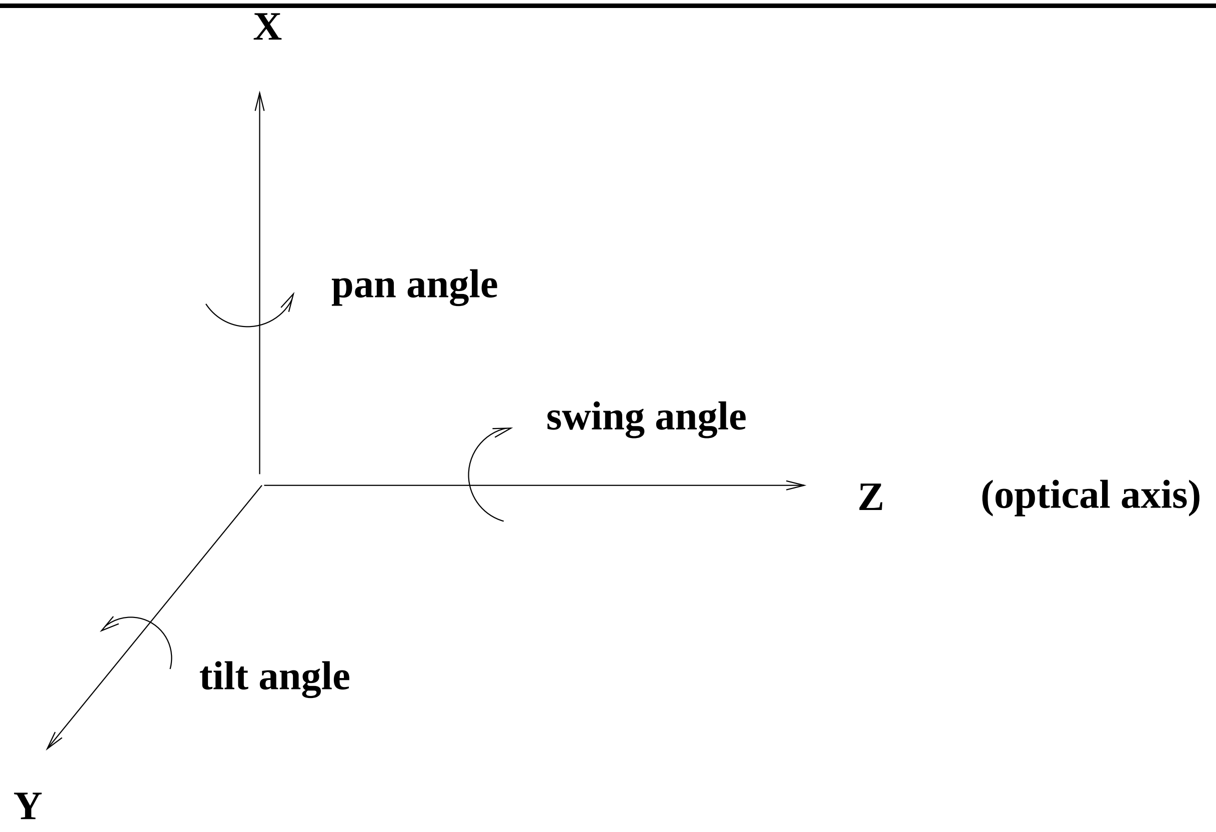
Rotation Matrix Representation: Euler angles

Assume rotation matrix R results from successive Euler rotations of the camera frame around its X axis by ω, its once rotated Y axis by ϕ, and its twice rotated Z axis by κ, then

$$R(\omega, \phi,\ \kappa) = R_X(\omega)R_Y(\phi)R_Z(\kappa)$$

where ω, ϕ, and κ are often referred to as pan, tilt, and swing angles respectively.

Rotation Matrix Representation: Euler angles



$$
\begin{aligned}
R_x(\omega) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix} \\
R_y(\phi) &= \begin{pmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{pmatrix} \\
R_z(\kappa) &= \begin{pmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Rotation Matrix: Rotation by a general axis

Let the general axis be $\omega = (\omega_x, \omega_y, \omega_z)$ and the rotation angle be θ. The rotation matrix R resulting from rotating around ω by θ can be expressed as

$$\begin{bmatrix} \cos\theta + \omega_x^2(1-\cos\theta) & \omega_x\omega_y(1-\cos\theta) - \omega_z\sin\theta & \omega_y\sin\theta + \omega_x\omega_z(1-\cos\theta) \\ \omega_z\sin\theta + \omega_x\omega_y(1-\cos\theta) & \cos\theta + \omega_y^2(1-\cos\theta) & -\omega_x\sin\theta + \omega_y\omega_z(1-\cos\theta) \\ -\omega_y\sin\theta + \omega_x\omega_z(1-\cos\theta) & \omega_x\sin\theta + \omega_y\omega_z(1-\cos\theta) & \cos\theta + \omega_z^2(1-\cos\theta) \end{bmatrix}$$

Rodrigues' rotation formula gives an efficient method for computing the rotation matrix.

Quaternion Representation of R

The relationship between a quaternion $q = [q_0, q_1, q_2, q_3]$ and the equivalent rotation matrix is

$$R = \begin{pmatrix} q_0q_0 + q_1q_1 - q_2q_2 - q_3q_3 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) \\ 2(q_2q_1 + q_0q_3) & q_0q_0 - q_1q_1 + q_2q_2 - q_3q_3 & 2(q_2q_3 - q_0q_1) \\ 2(q_3q_1 - q_0q_2) & 2(q_3q_2 + q_0q_1) & q_0q_0 - q_1q_1 - q_2q_2 + q_3q_3 \end{pmatrix}.$$

Here the quaternion is assumed to have been scaled to unit length, i.e., $|q| = 1$.

The axis/angle representation ω/θ is strongly related to a quaternion, according to the formula

$$\begin{pmatrix} \cos(\theta/2) \\ \omega_x \sin(\theta/2) \\ \omega_y \sin(\theta/2) \\ \omega_z \sin(\theta/2) \end{pmatrix}$$

where $\omega = (\omega_x, \omega_y, \omega_z)$ and $|\omega| = 1$.

R's Orthnormality

The rotation matrix is an orthnormal matrix, which means its rows (columns) are normalized to one and they are orthonal to each other. The orthnormality property produces

$R^t = R^{-1}$

Interior Camera Parameters

Parameters (c_0, r_0), s_x, s_y, and f are collectively referred to as *interior camera parameters*. They do not depend on the position and orientation of the camera. Interior camera parameters allow us to perform metric measurements, i.e., to convert pixel measurements to inch or mm.

Exterior Camera Parameters

Parameters like Euler angles ω, ϕ, κ, t_x, t_y, and t_z are collectively referred to as *exterior camera parameters*. They determine the position and orientation of the camera.

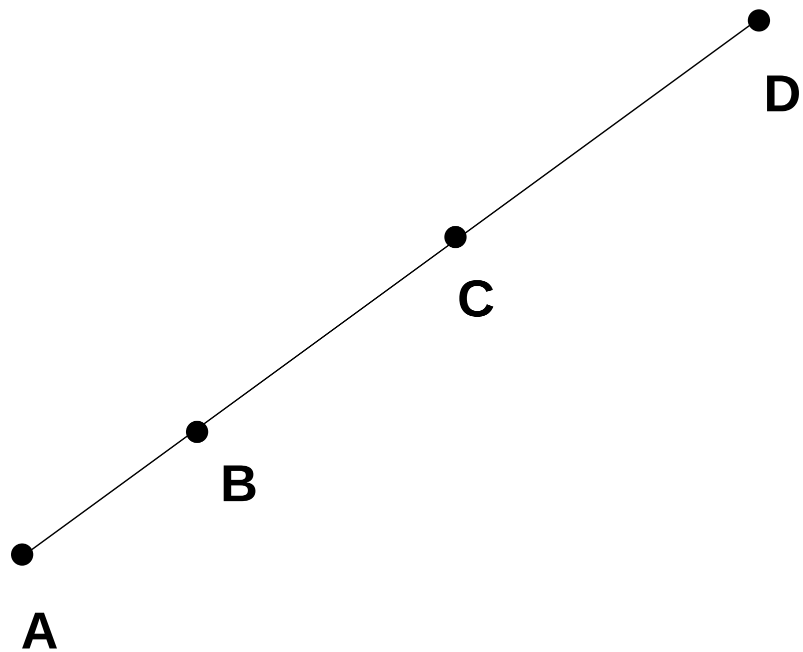
Camera Calibration and Pose Estimation

The purpose of camera calibration is to determine intrinsic camera parameters: c_0, r_0, s_x, s_y, and f. Camera calibration is also referred to as interior orientation problem in photogrammetry.

The goal of pose estimation is to determine exterior camera parameters: ω, ϕ, κ, t_x, t_y, and t_z. In other words, pose estimation is to determine the position and orientation of the object coordinate frame relative to the camera coordinate frame or vice versus.

Perspective Projection Invariants

Distances and angles are invariant with respect to Euclidian transformation (rotation and translation). The most important invariant with respect to perspective projection is called *cross ratio*. It is defined as follows:

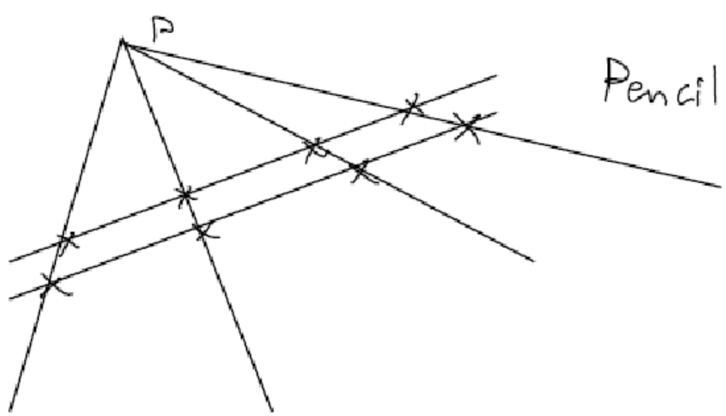


$$\tau(\mathbf{A,B,C,D}) = \frac{AC}{BC} \Big/ \frac{AD}{BD}$$

Cross-ratio is preserved under perspective projection.

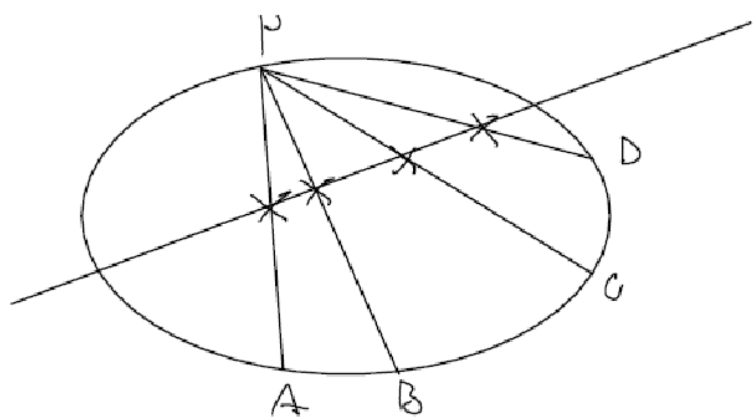
Projective Invariant for non-collinear points

Cross ratio of intersection points between a set of pencil of 4 lines and another line are only function of the angles among the pencil lines, independent of the intersection points on the lines. cross-ratio may be used for ground plane detection from multiple image frames.



Chasles' theorem:

Let A, B, C, D be distinct points on a (non-singular) conic (ellipse, circle, ..). If P is another point on the conic then the cross-ratio of intersections points on the pencil PA, PB, PC, PD does not depend on the point P. This means given A,B,C, and D, all points P on the same ellipse should satisfy Chasles's theorem. This theorem may be used for ellipse detection.



See section 19.3 and 19.4 of Daves book.