CONDITIONAL ENTROPY-CONSTRAINED VECTOR QUANTIZATION: HIGH-RATE THEORY and DESIGN ALGORITHMS *

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Abstract–**The performance of optimum vector quantizers subject to a conditional entropy constraint is studied in this paper. This new class of vector quantizers was originally suggested by Chou and Lookabaugh. A locally optimal design of this kind of vector quantizer can be accomplished through a generalization of the well known entropy-constrained vector quantizer (ECVQ) algorithm. This generalization of the ECVQ algorithm to a conditional entropy-constrained is called CECVQ, i.e., conditional ECVQ. Furthermore, we have extended the high-rate quantization theory to this new class of quantizers to obtain a new high-rate performance bound, which is a generalization of the works of Gersho and Yamada, Tazaki and Gray. The new performance bound is compared and shown to be consistent with bounds derived through conditional rate-distortion theory. Recently, a new algorithm for designing entropy-constrained vector quantizers was introduced by Garrido, Pearlman and Finamore, and is named entropy-constrained pairwise nearest neighbor (ECPNN). The algorithm is basically an entropy-constrained version of the pairwise nearest neighbor (PNN)**

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clustering algorithm of Equitz. By a natural extension of the ECPNN algorithm we develop another algorithm, called CECPNN, that designs conditional entropy-constrained vector quantizers. Through simulation results on synthetic sources, we show that CECPNN and CECVQ have very close distortion-rate performance. The advantages of CECPNN over CECVQ are that the CECPNN enables faster codebook design, and for the same distortion-rate performance the codebooks generated by the CECPNN tend to be smaller. We have compared the operational distortion-rate curves obtained by the quantization of synthetic sources using CECPNN codebooks with the analytical performance bound. Surprisingly, the theory based on the high-rate assumption seems to work very well for the tested synthetic sources at lower rates.

Index Terms - **Source coding, vector quantization, rate-distortion theory, information theory, entropy coding, clustering methods.**

1 Introduction

Let us consider an ergodic discrete-time random process $\{X_n\}_{n=1}^{\infty}$ described by an absolutely continuous probability density function. We will consider throughout that this process is *strict-sense stationary*, or in other words, none of the statistics is affected by a time shift. Let us define an $L-$dimensional vector taken from this information source described by a joint probability density function $f_{\mathbf{X}}(\mathbf{x})$, where $\mathbf{X} = [X_1, ..., X_L]^T \in \Re^L$, and $\Re^L$ is the $L-$dimensional Euclidean space. A memoryless vector quantizer (VQ) with block length L and average rate per vector performance measured by the output quantizer entropy $H(Q(\mathbf{X}))$ is composed by a collection of M reproduction vectors $\mathcal{C} = \{\mathbf{y}_1, ..., \mathbf{y}_M\} \subset \Re^L$, called the codebook, and by an encoder partition denoted $\mathcal{E}^{\mathcal{P}} = \{\mathcal{P}_1, ..., \mathcal{P}_M\}$ of $\Re^L$, where each $\mathcal{P}_i$ is a partition cell. The quantizer entropy is defined to be

$$H(Q(\mathbf{X})) = -\sum_{i=1}^{M} P_i \log P_i \tag{1}$$

where P_i is the prior probability that a source vector $\mathbf{X}$ is mapped to a particular codevector $\mathbf{y}_i$, and log(.) is the logarithm function defined on the Naperian base and the units of entropy are nats. Heretofore, the base of all logarithms is the Naperian base and the units of all entropies and measures of information are nats. The probability mass function P_i is measured by

$$P_i = \int_{\mathcal{P}_i} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}. \tag{2}$$

Associated with the codebook $\mathcal{C}$ and the encoder partition $\mathcal{E}^{\mathcal{P}}$ is the codeword length set $S = \{\ell_1, ..., \ell_M\}$, where each codeword ℓ_i is defined to be the self-entropy of each codevector, i.e.,

$$\ell_i = -\log P_i. \tag{3}$$

Finally, the memoryless vector quantizer Q is defined by

$$Q(\mathbf{x}) = \mathbf{y}_i \text{ if } \mathbf{x} \in \mathcal{P}_i. \tag{4}$$

Given a codebook $\mathcal{C}$ and a codeword set S the corresponding partition cell $\mathcal{P}_i$ (so called Voronoi region) can be defined by the biased nearest neighbor rule,

$$\mathcal{P}_i = \{\mathbf{x} : d(\mathbf{x}, \mathbf{y}_i) + \lambda\ell_i \leq d(\mathbf{x}, \mathbf{y}_m) + \lambda\ell_m; \forall m \neq i\} \tag{5}$$

where $d(.)$ is a general distortion criterion and the parameter λ is a Lagrange multiplier that reflects the relative importance of the codeword length and distortion. In this work we are interested in difference distortion measures of the form

$$d(\mathbf{x}, \mathbf{y}) = \rho(\parallel \mathbf{x} - \mathbf{y} \parallel) \tag{6}$$

where ρ is nondecreasing function of its argument and $\parallel . \parallel$ denotes a norm on $\Re^L$. The example widely used in rate-distortion theory is $\rho(\alpha) = \alpha^r$ for some $r \geq 1$. According to [1] such a distortion measure is called a *norm-based distortion measure*. Specifically, let $\parallel \mathbf{x} \parallel$ be a norm on $\Re^L$, that is,

$$\parallel \mathbf{x} \parallel \geq\ 0,\ \parallel a\mathbf{x} \parallel = |a| \parallel \mathbf{x} \parallel \ \text{ for } a \in \Re, \parallel \mathbf{x} \parallel = 0 \leftrightarrow \mathbf{x} = \mathbf{0},$$

and

$$\parallel \mathbf{x} + \mathbf{y} \parallel \leq \parallel \mathbf{x} \parallel + \parallel \mathbf{y} \parallel,$$

we can list the following examples of norms: the l_ν or Hölder norms defined by

$$\parallel \mathbf{x} \parallel_\nu = \left\{ \sum_{i=1}^{L} |x_i|^\nu \right\}^{1/\nu},$$

where $\nu = 2$ gives the Euclidean norm and $\nu = \infty$ the l_∞ norm of

$$\parallel \mathbf{x} \parallel_\infty = \lim_{\nu \to \infty} \parallel \mathbf{x} \parallel_\nu = \sup_i |x_i|.$$

Another difference distortion measure of interest is the weighted squared error measure defined as

$$\parallel \mathbf{x} - \mathbf{y} \parallel_{\mathbf{W}} = (\mathbf{x} - \mathbf{y})^T \mathbf{W} (\mathbf{x} - \mathbf{y}),$$

where T denotes transpose and $\mathbf{W}$ is a $L \times L$ symmetric positive definite matrix.

The average distortion D per vector obtained when the source vectors $\mathbf{X}$'s are mapped on codevectors $\mathbf{y}_i$'s is given by

$$D = E\{d(\mathbf{X}, Q(\mathbf{X}))\} = \sum_{i=1}^{M} \int_{\mathcal{P}_i} f_{\mathbf{X}}(\mathbf{x}) \rho(\| \mathbf{x} - \mathbf{y}_i \|) d\mathbf{x}. \tag{7}$$

The quantizer $Q(.)$ as defined is an entropy-constrained vector quantizer and its transmission rate R is its entropy when its outputs are entropy coded. The design of an entropy-constrained vector quantizer is generally based on the minimization of the functional

$$J = D + \lambda H(Q(\mathbf{X})). \tag{8}$$

We should observe that, if the entropy is not constrained the quantizer Q is called a level-constrained or minimum distortion vector quantizer, because the rate in nats/vector is allowed to reach its maximum of $R = \log M$. Note that in this situation each $\mathcal{P}_i$ is given through pure distortions comparisons in equation (5) with $\lambda = 0$. We should emphasize that we could use entropy coding in a level-constrained vector quantizer to operate at the rate of its entropy, but its average distortion D can be no smaller than the entropy-constrained quantizer in which minimum distortion is sought over all mapping rules achieving a particular entropy without fixing M.

One algorithm that finds a local minimum of distortion of the $M-$level quantizer has been proposed by Linde, Buzo and Gray [2] and is called generalized Lloyd algorithm (GLA). The design of GLA-quantizers is rather time-consuming. With the purpose of alleviating this problem, Equitz [3] proposed a recursive algorithm, called pairwise nearest neighbor (PNN), which is an adaptation to vector quantization of a hierarchical grouping procedure due to Ward [4]. The design of entropy-constrained vector quantizers using a modified GLA has been proposed by Chou, Lookabaugh and Gray [5] and is called ECVQ design algorithm. The algorithm approaches a local minimum of the distortion for a given entropy of the quantizer output. The equations (7) and (8) are the functionals that are respectively used in the minimization process of the design of memoryless VQ's carried out by the GLA and ECVQ design. Recently an entropy constrained version of the PNN design algorithm was proposed by Garrido, Pearlman and Finamore [6] and is called entropy-constrained pairwise nearest neighbor, or in short ECPNN.

A new class of non-memoryless vector quantizers was introduced recently by Chou and Lookabaugh [7]. The biased distortion measure used in the codebook design and the quantization (see eqn.(14)) was previously used in speech recognition by Huang and Gray [8]. These vector quantizers have the capability to exploit non-memoryless sources more efficiently than the mentioned vector quantizers. They named this new class as conditional entropy-constrained vector quantizers CECVQ's, and the algorithm that designs these vector quantizers is a modification of the ECVQ design. The functional used for minimization is defined by

$$\tilde{J} = E\{d(\mathbf{X}_k, Q(\mathbf{X}_k)\} + \lambda H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l)) \tag{9}$$

with,

$$\begin{aligned}
H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l)) &= -\sum_{k=1}^{M}\sum_{l=1}^{M} P_{lk}\log P_{k|l}, &(10)\\
P_{lk} &= \int_{\mathcal{P}_k}\int_{\mathcal{P}_l} f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{x}_l,\mathbf{x}_k)d\mathbf{x}_l d\mathbf{x}_k, &(11)\\
P_l &= \int_{\mathcal{P}_l} f_{\mathbf{X}_l}(\mathbf{x}_l)d\mathbf{x}_l, &(12)\\
P_{k|l} &= \frac{P_{lk}}{P_l} &(13)
\end{aligned}$$

where $H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l))$ is the first-order conditional block entropy, with $\mathbf{X}_k$ as the current L-dimensional vector to be encoded and $\mathbf{X}_l$ as the previous encoded L-dimensional vector. The quantization rule of CECVQ is non-memoryless since the partition cell $\mathcal{P}_i$ is defined by a distortion measure biased by a probability conditioned on the last reproduction, according to

$$\mathcal{P}_i = \{\mathbf{x}_k : d(\mathbf{x}_k, \mathbf{y}_i) - \lambda\log P_{i|l} \leq d(\mathbf{x}_k, \mathbf{y}_m) - \lambda\log P_{m|l}; \forall m \neq i\}. \tag{14}$$

When the number of quantizer cells is finite, the conditional block entropy-constrained vector quantizer as defined can be recognized as a finite-state machine, where the states are the quantizer cells. Through the above quantizer rule, the expected distortion $D = E\{\mathbf{X}_k, Q(\mathbf{X}_k)\}$ is minimized subject to a constraint on the first-order conditional block entropy $H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l))$. The codevector $Q(\mathbf{X}_k)$ at time k is noiselessly encoded according to a conditional entropy matched to $P_{k|l}$ with $l = k-1$, which changes according to the realization of the previous codevector $Q(\mathbf{X}_l)$.

The final remark is that in this work we are mainly interested in optimal VQ's but we should point out that conditional entropy constrained VQ's could be designed based on lattice quantizers. Lattice quantizers have partitions defined by a regular array that covers the L-dimensional space uniformly. For example in one dimension a lattice quantizer has a line segment as a geometrical shape of a partition, and in two dimensions the partition shape can be hexagons, squares or other polygons. For a detailed treatise on lattices, see [9].

In this paper we present a theoretical study of this class of vector quantizers using high rate asymptotic theory. With this tool we are able to better understand the performance of such schemes.

In Section II, the high rate entropy-constrained theory is presented as a prelude to our generalization to the case when the quantization of the current vector is conditioned on the reproduction of the previous vector of the same dimension. The link between the high-rate quantization theory and rate-distortion theory is presented in Section III. In Section IV, we present analytical and numerical results which quantify the advantage in performance of conditional entropy-constrained vector quantization compared to memoryless entropy-constrained vector quantization as a function of the memory between adjacent source blocks. In Section V, we present a modification of the ECPNN algorithm in order to design conditional entropy-constrained vector quantizers and compare its results with CECVQ for synthetic data. Finally, in Section VI, we give our conclusions and directions for future research.

2 Constrained Conditional Entropy High Rate Quantizer Bound

In this section, we derive a lower bound to the asymptotic average distortion D for a given constraint of conditional entropy quantizer output. This result is an extension of the work of Gersho [10] and Yamada, Tazaki and Gray [1]. Here is used the approach given by Yamada *et al.*, where the partitioning $\mathcal{P}_i$ of $\Re^L$ space is done by $L-$dimensional spheres in contrast to L-dimensional *optimal polytopes* defined by Gersho. Optimal polytopes are generally unknown and are known in the case of the mean-square error just for $L \leq 3$. Nevertheless the main results of Yamada *et al.* are connected with those of Gersho's. First, we shall present some results on the *high rate quantization theory.* For an excellent review of this topic the reader is directed to Chapter 5 in [11].

2.1 High-Rate Quantization Theory

High-rate quantization theory is a useful tool for determining the performance of optimum vector quantizers. This theory assumes that the number of quantization levels M is large enough and the probability density function $f_{\mathbf{X}}(\mathbf{x})$ is sufficiently smooth to be nearly constant within all bounded partition cells $\mathcal{P}_i$, i.e.,

$$f_{\mathbf{X}}(\mathbf{x}) \sim f_{\mathbf{X}}(\mathbf{y}_i) \text{ if } \mathbf{x} \in \mathcal{P}_i.$$

where $\mathbf{y}_i$ is the $i-$th reconstruction vector. Consequently, we can write the probability $P_i = \Pr[\mathbf{X} \in \mathcal{P}_i]$ as

$$P_i = \int_{\mathcal{P}_i} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \approx f_{\mathbf{X}}(\mathbf{y}_i) V(\mathcal{P}_i),$$

where $V(\mathcal{P}_i)$ is the infinitesimal $L-$dimensional volume,

$$V(\mathcal{P}_i) = \int_{\mathcal{P}_i} d\mathbf{x}.$$

Therefore, we are able to write $f_{\mathbf{X}}(\mathbf{x}) \approx \frac{P_i}{V(\mathcal{P}_i)}$ if $\mathbf{x} \in \mathcal{P}_i$. Consider now the distortion integral,

$$D = \sum_{i=1}^{M} \int_{\mathcal{P}_i} f_{\mathbf{X}}(\mathbf{x}) \rho(\| \mathbf{x} - \mathbf{y} \|) d\mathbf{x}.$$

Assuming negligible contribution to unbounded partition cells and the approximations above, we obtain

$$D \approx \sum_{i=1}^{M} \frac{P_i}{V(\mathcal{P}_i)} \int_{\mathcal{P}_i} \rho(\| \mathbf{x} - \mathbf{y}_i \|) d\mathbf{x}. \tag{15}$$

This expression corresponds to the distortion due to source vectors that are mapped onto $\mathbf{y}_i$'s contained in bounded partition cells, or what is also called granular distortion.

Another important concept that is useful for the performance bound derivation is the so called *codevector point density function*, $\gamma(\mathbf{x})$. Let us define γ as a continuous smooth function, with the following characteristics:

$$\int \gamma(\mathbf{x}) d\mathbf{x} = 1;$$

and the integral of γ over a region $\mathcal{R}$ times the size of the codebook M produces the number of codevectors, η, in $\mathcal{R}$, i.e.

$$\eta(\mathcal{R}) = M \int_{\mathcal{R}} \gamma(\mathbf{x}) d\mathbf{x}.$$

For M large enough, so that $\gamma(.)$ is approximately constant in a partition $\mathcal{P}_i$ containing one codevector

$$\eta(\mathcal{P}_i) = 1 = M \int_{\mathcal{P}_i} \gamma(\mathbf{x}) d\mathbf{x} \approx M\gamma(\mathbf{y}_i) V(\mathcal{P}_i)$$

and consequently,

$$V(\mathcal{P}_i) = \frac{1}{M\gamma(\mathbf{y}_i)}. \tag{16}$$

The high rate expression for the distortion, (15), and the codevector point density equation, (16), are used in the derivation of a lower bound to the the average distortion obtained by considering the optimum partition cell $\mathcal{P}_i$ without regard to whether or not a quantizer can have such partitioning. Yamada *et al.* [1] shows that the optimal partition shape, i.e., in the sense of minimization of the distortion expression, (15), is an L-dimensional sphere with respect to the norm used to define the distortion measure. More precisely, Yamada *et al.* derive the following lower bound to the minimum distortion of a level constrained vector quantizer,

$$\hat{D}^{\mathrm{LC}} \geq \hat{D}^{\mathrm{LC}}_{\mathcal{L}} \geq \hat{D}_{lb} = \frac{L}{L+r} V_L^{-r/L} E\{(M\gamma(\mathbf{X}))^{-r/L}\} \tag{17}$$

where $\hat{D}^{\rm LC}$ denotes the operational distortion-rate performance per vector for a level constrained vector quantizer, $\hat{D}^{\rm LC}_{\mathcal{L}}$ the respective lower bound, r is the power to which the norm is raised, and V_L is the volume of the unit sphere. The unit sphere is defined by the set $\{\mathbf{x} : \| \mathbf{x} \| \leq 1\}$ where $\| \, . \, \|$ denotes the given norm. Consequently, V_L is measured by

$$V_L = \int_{\mathbf{x}:\|\mathbf{x}\|\leq 1} d\mathbf{x}$$

and for example, according to [1] for l_ν norms or the ν-th power distortion measure

$$V_L = V_L(\nu) = \frac{2^L(\Gamma(1/\nu))^L}{L\Gamma(L/\nu)\nu^{L-1}}, \tag{18}$$

for l_∞ norm

$$V_L = 1,$$

and for the weighted squared error distortion measure

$$V_L = (\det \mathbf{W})^{-1/2} V_L(\nu), \text{ with } \nu = 2,$$

where $\Gamma(.)$ denotes the usual gamma function. Note that if we divide equation (17) by L we obtain the operational distortion-rate performance per sample, denoted $\hat{D}^{\rm LC}_L$.

In equation (17), it can be shown ([1] and [10]) that the rightmost equality is achieved iff

$$\gamma(\mathbf{x}) = \frac{f_{\mathbf{X}}(\mathbf{x})^{L/(L+r)}}{\| f_{\mathbf{X}}(\mathbf{x}) \|_{L/(L+r)}^{L/(L+r)}}$$

where here the norm $\| \, . \, \|$ denotes the L_p norm. Next we develop the *constrained conditional lower bound.*

2.2 Performance Bound

The quantizer output conditional block entropy given by

$$H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l)) = H(Q(\mathbf{X}_l), Q(\mathbf{X}_k)) - H(Q(\mathbf{X}_l)), \tag{19}$$

is the rate of the quantizer. When the number of partition cells M is large and hence the rate is large, let us assume that

$$f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{x}_l, \mathbf{x}_k) \sim f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l, \mathbf{y}_k) \text{ for } (\mathbf{x}_l, \mathbf{x}_k) \in \mathcal{P}_l \times \mathcal{P}_k, \tag{20}$$

$$f_{\mathbf{X}_l}(\mathbf{x}_l) \sim f_{\mathbf{X}_l}(\mathbf{y}_l) \text{ for } \mathbf{x}_l \in \mathcal{P}_l \tag{21}$$

where $\times$ denotes *Cartesian product* and $\mathbf{y}$'s are the L-dimensional reproduction levels. Here the assumption is that the joint and marginal densities are slowly varying functions on the partition cells. We should have in mind that in high rates the partition cells can be represented by infinitesimal orthotopes (orthogonal figures) in such a way that Cartesian products between two infinitesimal orthotopes in L dimensions generates a $2L$-dimensional infinitesimal orthotope. Therefore, by our previous assumptions

$$\int_{\mathcal{P}_l}\int_{\mathcal{P}_k} d\mathbf{x}_l d\mathbf{x}_k = \left(\int_{\mathcal{P}_l} d\mathbf{x}_l\right)\left(\int_{\mathcal{P}_k} d\mathbf{x}_k\right) = V(\mathcal{P}_l)V(\mathcal{P}_k)$$

and finally obtain, based on equations (11) and (12), the probability mass functions,

$$P_{lk} = f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l,\mathbf{y}_k)V(\mathcal{P}_l)V(\mathcal{P}_k), \tag{22}$$

$$P_l = f_{\mathbf{X}_l}(\mathbf{y}_l)V(\mathcal{P}_l), \tag{23}$$

of the partition cells $\{\mathcal{P}_l \times \mathcal{P}_k\}$ and $\mathcal{P}_l$, respectively, where we have adopted the equality sign in the above equations just for convenience. As was discussed, $V(\mathcal{P}_k) = \frac{1}{M\gamma(\mathbf{y}_k)}$ and $V(\mathcal{P}_l) = \frac{1}{M\gamma(\mathbf{y}_l)}$ and consequently

$$P_{lk} = \frac{f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l,\mathbf{y}_k)}{M^2\gamma(\mathbf{y}_l)\gamma(\mathbf{y}_k)}, \tag{24}$$

$$P_l = \frac{f_{\mathbf{X}_l}(\mathbf{y}_l)}{M\gamma(\mathbf{y}_l)}. \tag{25}$$

At this point some interpretation can be done. Note that,

$$M^2\gamma(\mathbf{y}_l)\gamma(\mathbf{y}_k)V(\mathcal{P}_l)V(\mathcal{P}_k) \approx \int_{\mathcal{P}_k}\int_{\mathcal{P}_l} M^2\gamma(\mathbf{x}_l)\gamma(\mathbf{x}_k)d\mathbf{x}_l d\mathbf{x}_k = \eta(\{\mathcal{P}_l \times \mathcal{P}_k\}) = 1 \tag{26}$$

represents the number of codevector points in the region generated by $\{\mathcal{P}_l \times \mathcal{P}_k\}$ that is of course equal to one. If we define a function $\zeta(\mathbf{x}_l,\mathbf{x}_k) = \gamma(\mathbf{x}_l)\gamma(\mathbf{x}_k)$, it can be interpreted as a joint codevector point density function, which in the case of a CECVQ can be considered separable, because of the high rate quantization assumption.

Considering now the first order conditional block entropy, substituting (24) and (25) in (19),

$$\begin{aligned} H \approx & - \sum_{k=1}^{M}\sum_{l=1}^{M} f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l,\mathbf{y}_k)V(\mathcal{P}_l)V(\mathcal{P}_k)\log\frac{f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l,\mathbf{y}_k)}{M^2\gamma(\mathbf{y}_l)\gamma(\mathbf{y}_k)} \\ & + \sum_{l=1}^{M} f_{\mathbf{X}_l}(\mathbf{y}_l)V(\mathcal{P}_l)\log\frac{f_{\mathbf{X}_l}(\mathbf{y}_l)}{M\gamma(\mathbf{y}_l)} \end{aligned} \tag{27}$$

which, after some algebra, reduces to

$$\begin{aligned}H \approx & - \sum_{k=1}^{M}\sum_{l=1}^{M} f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l,\mathbf{y}_k)\log\left(\frac{f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l,\mathbf{y}_k)}{f_{\mathbf{X}_l}(\mathbf{y}_l)}\right)V(\mathcal{P}_l)V(\mathcal{P}_k) \\ & + \sum_{k=1}^{M}\sum_{l=1}^{M} f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{y}_l,\mathbf{y}_k)\log(M\gamma(\mathbf{y}_k))V(\mathcal{P}_l)V(\mathcal{P}_k).\end{aligned}$$

When sums are approximated by integrals, we obtain

$$H \approx -\int\int f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{x}_l,\mathbf{x}_k)\log f_{\mathbf{X}_k|\mathbf{X}_l}(\mathbf{x}_k|\mathbf{x}_l)d\mathbf{x}_l d\mathbf{x}_k - E\left\{\log\frac{1}{M\gamma(\mathbf{X}_k)}\right\}$$

or finally,

$$H \approx h(\mathbf{X}_k|\mathbf{X}_l) - LE\left\{\log\left(\frac{1}{M\gamma(\mathbf{X}_k)}\right)^{1/L}\right\}. \tag{28}$$

where

$$h(\mathbf{X}_k|\mathbf{X}_l) = -\int\int f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{x}_l,\mathbf{x}_k)\log f_{\mathbf{X}_k|\mathbf{X}_l}(\mathbf{x}_k|\mathbf{x}_l)d\mathbf{x}_l d\mathbf{x}_k$$

is the conditional differential entropy of the current block $\mathbf{X}_k$ given the previous block $\mathbf{X}_l$. Now, consider Jensen's inequality, $E\{\phi(\mathbf{X})\} \geq \phi(E\{\mathbf{X}\})$, where ϕ is a convex cup function and strictly equality holds iff $\mathbf{X}$ has a uniform distribution. Applying this inequality, to the convex cup function $-\log$, the high rate conditional entropy expression is lower bounded according to

$$H \geq h(\mathbf{X}_k|\mathbf{X}_l) - L\log\left(E\left\{\left(\frac{1}{M\gamma(\mathbf{X}_k)}\right)^{1/L}\right\}\right). \tag{29}$$

Since H is the conditional block entropy in (19), (29) implies

$$E\left\{\left(\frac{1}{M\gamma(\mathbf{X}_k)}\right)^{1/L}\right\} \geq e^{-\frac{1}{L}(H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l))-h(\mathbf{X}_k|\mathbf{X}_l))}. \tag{30}$$

Now referring to the equation (17), since x^r is a convex cup function of x for $r \geq 1$, we can apply Jensen's inequality to the lower bound $\hat{D}_{lb}$ to obtain another lower bound

$$\hat{D}_{lb} \geq \frac{L}{L+r}(V_L)^{-r/L}\left(E\left\{\left(\frac{1}{M\gamma(\mathbf{X}_k)}\right)^{1/L}\right\}\right)^r. \tag{31}$$

Note that the expression above is valid for a conditional entropy-constrained vector quantizer because the distortion integral depends only on the vector to be encoded. Now, let us combine the bound in (17) with (31) and (30) to obtain

$$\hat{D}^{\text{CE}} \geq \hat{D}^{\text{CE}}_{\mathcal{L}} \geq L\hat{D}^{\text{CE}}_{lb,L} = \frac{L}{L+r}(V_L)^{-r/L}e^{-\frac{r}{L}(H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l))-h(\mathbf{X}_k|\mathbf{X}_l))}, \tag{32}$$

and, according to Jensen's inequality, the rightmost equality is achieved iff $\gamma(\mathbf{x})$ is a constant. This condition is satisfied by high rate lattice vector quantizers because the reproduction vectors are uniformly distributed over some set having probability 1. Therefore since $\gamma(\mathbf{x})$ must have unit integral over the granular region $\mathcal{P}_\mathcal{G}$ with volume V

$$\gamma(\mathbf{x}) = \frac{1}{V}\mathcal{I}_{\mathcal{P}_\mathcal{G}}(\mathbf{x}) \tag{33}$$

where $\mathcal{I}$ denotes the indicator function.

The high-rate performance of an entropy-constrained vector quantizer per vector is given according to [1]:

$$\hat{D}^{\mathrm{E}} \geq \hat{D}^{\mathrm{E}}_{\mathcal{L}} \geq L\hat{D}^{\mathrm{E}}_{lb,L} = \frac{L}{L+r}(V_L)^{-r/L}e^{-\frac{r}{L}(H(Q(\mathbf{X}_k))-h(\mathbf{X}_k))}. \tag{34}$$

The quantity $h(\mathbf{X}_k)$ above is the (unconditional) differential entropy of the block $\mathbf{X}_k$.

Consider now the analytical lower bounds $\hat{D}^{\mathrm{CE}}_{lb,L}$ and $\hat{D}^{\mathrm{E}}_{lb,L}$ in (32) and (34) at the same quantizer rate $R = \frac{1}{L}H(Q(\mathbf{X}_k)) = \frac{1}{L}H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l))$. Since $h(\mathbf{X}_k|\mathbf{X}_l) \leq h(\mathbf{X}_k)$, we conclude that

$$\hat{D}^{\mathrm{CE}}_{lb,L} \leq \hat{D}^{\mathrm{E}}_{lb,L}. \tag{35}$$

Equality holds if and only if the source vectors are statistically independent.

The constrained conditional entropy high rate quantizer lower bound generalizes the constrained entropy high rate quantizer lower bound as should be expected. Note, that the bounds (17) and (34) and (35) are valid for l_∞ norm and weighted squared error distortion measure.

At a given average distortion for each system, the conditional entropy $\hat{R}^{\mathrm{CE}}_L \leq \hat{R}^{\mathrm{E}}_L$, the unconditional entropy. Therefore, at any rate R, the actual average distortions are consistent in relationship to their bounds, i.e., $\hat{D}^{\mathrm{CE}}_L \leq \hat{D}^{\mathrm{E}}_L$.

Our objective now is to compare a $2L$ memoryless entropy VQ with an L non-memoryless conditional entropy VQ, since both methods require access to the same set of source variables. Let us define now for the l_ν norms the spherical quantization coefficient as:

$$C^{sp}(L,r,\nu) = \frac{1}{L+r}(V_L(\nu))^{-r/L}. \tag{36}$$

In Figure 1 is plotted the behavior of the spherical quantization coefficient when we increase the dimension L for a fixed r and ν. Clearly, the function is monotonically increasing or decreasing in L, depending on r and ν parameters. Note that for the case that $r > \nu$, the $C^{sp}(L,r,\nu)$ is a non-decreasing function of the dimension L. For this case, the distortion increases with the vector dimension L contrary to principles of information theory (see Lookabaugh and Gray in [12]). Consequently we shall disregard the case $r > \nu$.

Let us assume that $\nu = r$. In this case the spherical quantization coefficient is monotonically decreasing function in L or

$$C^{sp}(L, r, r) > C^{sp}(L + \delta, r, r) \ \forall \ r \geq 1 \text{ and } L \geq 1$$

with $\delta > 0$. Note that this case is perhaps the most important one, because with the constraint $\nu = r$ the norm-based distortion measure is additive in a single letter fashion.

The analytical lower bounds to be compared are:

$$\begin{aligned} \hat{D}^{\text{CE}}_{lb,L} &= C^{sp}(L, r, r) e^{-\frac{r}{L}(H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l)) - h(\mathbf{X}_k|\mathbf{X}_l)))}, \\ \hat{D}^{\text{E}}_{lb,2L} &= C^{sp}(2L, r, r) e^{-\frac{r}{2L}(H(Q(\mathbf{X}_l,\mathbf{X}_k)) - h(\mathbf{X}_l,\mathbf{X}_k))}. \end{aligned} \tag{37}$$

at the same rate $R = \frac{1}{L}H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l)) = \frac{1}{2L}H(Q(\mathbf{X}_l, \mathbf{X}_k))$.

We shall derive conditions under which L-vector conditional entropy VQ is superior to $2L$-vector unconditional VQ. We shall assume that the bounds $\hat{D}^{\text{E}}_{lb,L}$ and $\hat{D}^{\text{CE}}_{lb,L}$ have the same degree of multiplicative tightness to their actual functions, $\hat{D}^{\text{E}}_{L}$ and $\hat{D}^{\text{CE}}_{L}$, respectively, since they were derived in exactly the same way with just the substitution of conditional entropy for entropy. This assumption leads to

$$\frac{\hat{D}^{\text{E}}_{2L}}{\hat{D}^{\text{CE}}_{L}} = \frac{\hat{D}^{\text{E}}_{lb,2L}}{\hat{D}^{\text{CE}}_{lb,L}}. \tag{38}$$

Therefore, by showing conditions under which

$$\frac{\hat{D}^{\text{E}}_{lb,2L}}{\hat{D}^{\text{CE}}_{lb,L}} \geq 1,$$

we can infer the superiority of the conditional L-vector ECVQ over the unconditional $2L$-vector ECVQ. Forming the ratio of the distortions in (37), equating the rates to

$$R = \frac{1}{2L}H(Q(\mathbf{X}_k, \mathbf{X}_l)) = \frac{1}{L}H(Q(\mathbf{X}_k)|Q(\mathbf{X}_l)), \tag{39}$$

and using the formulas for the spherical quantization coefficients in (36) and the unit volume in (18), properties of gamma functions [13, eq. 6.1.18], and information-theoretic relationships, produce the following expression

$$\frac{\hat{D}^{\text{E}}_{lb,2L}}{\hat{D}^{\text{CE}}_{lb,L}} = G(L/r) \cdot \exp(\frac{r}{2L}I(\mathbf{X}_k; \mathbf{X}_l)) \tag{40}$$

with $I(\mathbf{X}_k;\mathbf{X}_l)$ denoting the average mutual information betwwen the source blocks and

$$G(\alpha) = (\frac{\alpha+1}{\alpha+\frac{1}{2}})(\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha+1)})^{1/2\alpha}, \alpha \geq 0. \tag{41}$$

Use has been made of the following relations for stationary sources,

$$\begin{aligned} I(\mathbf{X}_k;\mathbf{X}_l) &= \frac{1}{2}h(\mathbf{X}_k,\mathbf{X}_l) - h(\mathbf{X}_k|\mathbf{X}_l) \\ &= h(\mathbf{X}_k) - h(\mathbf{X}_k|\mathbf{X}_l) \end{aligned} \tag{42}$$

Therefore, $\frac{\hat{D}^{\mathrm{E}}_{lb,2L}}{\hat{D}^{\mathrm{CE}}_{lb,L}} \geq 1$ if

$$\frac{1}{L}I(\mathbf{X}_k;\mathbf{X}_l) \geq \frac{2}{r}\ln\frac{1}{G(L/r)}. \tag{43}$$

The function $G(\alpha)$ is plotted in Fig. 2 for positive values of α. The highest required per-letter average mutual information or memory between the L-vectors occurs at the feasible minimum of 0.939 of the function G, where α equals 2 and is rational. One could use this minimum to develop a greatest lower bound on memory in the form

$$\frac{1}{L}I(\mathbf{X}_k;\mathbf{X}_l) \geq \frac{2}{r}(.0629). \tag{44}$$

In the limit as $L \longrightarrow \infty$ for fixed finite $r, G(L/r) \longrightarrow 1$ and $\frac{1}{L}I(\mathbf{X}_k;\mathbf{X}_l) \longrightarrow 0$, so that

$$\lim_{L\to\infty} \hat{D}^{\mathrm{E}}_{2L} = \lim_{L\to\infty} \hat{D}^{\mathrm{CE}}_{L}.$$

Since the per letter memory tends to zero in the limit, the conditional entropy coding can gain no advantage. We shall return in a later section to the quantification of the distortion ratio in (40) as a function of source memory for finite L.

3 Comparison with Rate-Distortion Theory

Rate-distortion theory provides the unbeatable bounds for compression of information sources. In this section we compare the results of the last section with the rate-distortion function of a given stationary process, and draw analogies between the constrained conditional entropy quantization performance bound and the conditional rate-distortion function.

3.1 An Expected Comparison

In particular, we are interested in the rate-distortion function $R(D)$ for stationary sources with memory. According to Gray [11, 102-103], the rate-distortion function is defined by

$$R(D) = \lim_{L\to\infty} R_L(D), \tag{45}$$

with

$$
\begin{aligned}
R_L &= L^{-1} \inf_{f \in \mathcal{F}_D} I(\mathbf{X};\mathbf{Y}),\\
f &\stackrel{\text{def}}{=} f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})\\
\mathcal{F}_D &= \left\{ f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) : \int\int f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d_L(\mathbf{x},\mathbf{y})\, d\mathbf{x}d\mathbf{y} \le D \right\},\\
I(\mathbf{X},\mathbf{Y}) &= \int\int f_{\mathbf{X}}(\mathbf{x})\, f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x}) \log \frac{f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})}{f_{\mathbf{Y}}(\mathbf{y})} d\mathbf{x}d\mathbf{y},
\end{aligned}
$$

where $f_{\mathbf{Y}|\mathbf{X}}(\mathbf{y}|\mathbf{x})$ is the conditional probability density for reproducing vectors given source vectors. Note that we have normalized the rate and distortion by L in the equations above. The rate-distortion performance $R(D)$ is well defined because it can be shown for stationary sources that the limit in (45) always exists. With the exception of the Gaussian source, $R_L(D)$ is generally very difficult to evaluate, but it can be bounded in the case of difference-distortion measures by the so called *vector Shannon lower bound* if it exists [11, pp. 106-111]. The vector Shannon lower bound is given by:

$$R_{SLB}(D) = h(\mathbf{X}) + \log a(D) - Db(D)$$

where $a(D)$ and $b(D)$ are the solutions of

$$a(D) \int_{\Re^L} e^{-b(D)\rho(\|\mathbf{x}\|)} d\mathbf{x} = 1,$$

$$a(D) \int_{\Re^L} \rho(\| \mathbf{x} \|) e^{-b(D)\rho(\|\mathbf{x}\|)} d\mathbf{x} = D.$$

Note that in the equation above, $\rho(.)$ is not normalized in relation to the vector size L. Yamada *et al.* have shown for norm-based distortion measures of the class $\rho(x) = x^r$ with $r \ge 1$, that the vector Shannon lower bound in nats per vector is given by

$$R^{SLB}(D) = h(\mathbf{X}_k) - \frac{L}{r}\log(D/L) - \log V_L - \frac{L}{r}\log(er\Gamma(1+L/r)^{r/L}). \tag{46}$$

Now normalizing the rate and distortion by the dimension L, we have

$$R_L(D) \ge R_L^{SLB}(D) = \frac{1}{L}h(\mathbf{X}_k) - \frac{1}{r}\log(D) - \frac{1}{L}\log V_L - \frac{1}{r}\log(er\Gamma(1+L/r)^{r/L}). \tag{47}$$

According to Lin'kov [14], under smoothness constraints on the joint probability density function, $R_L(D) = R_L^{SLB}(D)$, for $D \le D_\epsilon$, where D_ϵ corresponds to a small distortion.

Let us consider the behavior of the high-rate quantization bounds when the vector size $L \to \infty$. The operational rate-distortion performance per sample of a conditional entropy-constrained VQ is given by the inversion of (32) with the normalization of D

$$\hat{R}_L^{\mathrm{CE}}(D) \geq \hat{R}_{lb,L}^{\mathrm{CE}}(D) = \frac{1}{L} h(\mathbf{X}_k|\mathbf{X}_l) - \frac{1}{r}\log(D) - \frac{1}{L}\log V_L - \frac{1}{r}\log(r+L). \tag{48}$$

Now taking the difference between the quantizer lower bound and the Shannon bound, we obtain

$$\Delta = (\hat{R}_{lb,L}^{\mathrm{CE}}(D) - R_L(D)) = \frac{1}{L}(h(\mathbf{X}_k|\mathbf{X}_l) - h(\mathbf{X}_k)) + \frac{1}{L}\log\left[\left(\frac{e}{1+\frac{L}{r}}\right)^{(L/r)} \Gamma\left(1+\frac{L}{r}\right)\right] \tag{49}$$

For large L, it is shown in [1] that, using Stirling's approximation,

$$\log\left[\left(\frac{e}{1+\frac{L}{r}}\right)^{(L/r)} \Gamma\left(1+\frac{L}{r}\right)\right] \approx \frac{\sqrt{2\pi}}{e}\left(1+\frac{L}{r}\right)^{1/2}.$$

Now evaluating the limits

$$\lim_{L\to\infty} \Delta = \lim_{L\to\infty} \frac{1}{L}[h(\mathbf{X}_k|\mathbf{X}_l) - h(\mathbf{X}_k)] + \lim_{L\to\infty}\left[\frac{1}{L}\log\left(\frac{\sqrt{2\pi}}{e}\right) + \frac{1}{2L}\log\left(1+\frac{L}{r}\right)\right] = 0,$$

Since the Shannon bound $R_L^{SLB}(D) = R_L(D)$ for $D \leq D_\epsilon$, the quantizer lower bound, $\hat{R}_{lb,L}^{\mathrm{CE}}(D)$ approaches $R(D)$ in the limit as L tends to infinity in a region of low distortion or high rate.

3.2 Analogies with the Conditional Rate-Distortion Function

In this section, we want to highlight the connection between the high-rate quantization theory and rate-distortion theory via the conditional rate-distortion function. This branch of the rate-distortion theory was developed mainly by Gray in [15] and [16]. By definition the $L-$th order conditional rate-distortion function of a process with memory is described by:

$$R_L^{(\mathbf{X}_k|\mathbf{X}_l)}(D) = L^{-1} \inf_{f \in \mathcal{F}_D} I(\mathbf{X}_k; \mathbf{Y}|\mathbf{X}_l), \tag{50}$$

$$f \stackrel{\text{def}}{=} f_{\mathbf{Y}|\mathbf{X}_k\mathbf{X}_l}(\mathbf{y}|\mathbf{x}_k, \mathbf{x}_l)$$

$$\mathcal{F}_D = \left\{ f_{\mathbf{Y}|\mathbf{X}_k\mathbf{X}_l}(\mathbf{y}|\mathbf{x}_k,\mathbf{x}_l) : \int\int\int f_{\mathbf{Y}|\mathbf{X}_k\mathbf{X}_l}(\mathbf{y}|\mathbf{x}_k,\mathbf{x}_l) f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{x}_l,\mathbf{x}_k)\, d_L(\mathbf{x}_k,\mathbf{y})\, d\mathbf{x}_k d\mathbf{y} d\mathbf{x}_l \leq D \right\},$$

$$I(\mathbf{X}_k;\mathbf{Y}|\mathbf{X}_l) = \int\int\int f_{\mathbf{X}_l\mathbf{X}_k}(\mathbf{x}_l,\mathbf{x}_k)\, f_{\mathbf{Y}|\mathbf{X}_k\mathbf{X}_l}(\mathbf{y}|\mathbf{x}_k,\mathbf{x}_l) \log \frac{f_{\mathbf{X}_k\mathbf{Y}|\mathbf{X}_l}(\mathbf{x}_k,\mathbf{y}|\mathbf{x}_l)}{f_{\mathbf{X}_k|\mathbf{X}_l}(\mathbf{x}_k|\mathbf{x}_l) f_{\mathbf{Y}|\mathbf{X}_l}(\mathbf{y}|\mathbf{x}_l)} d\mathbf{x}_k d\mathbf{y} d\mathbf{x}_l,$$

where $f_{\mathbf{Y}|\mathbf{X}_k\mathbf{X}_l}(\mathbf{y}|\mathbf{x}_k,\mathbf{x}_l)$ is the conditional probability density for reproducing vectors given source vectors. The conditional rate-distortion function can be interpreted as the rate of a source $\mathbf{X}_k$ given $\mathbf{X}_l$ subject to a fidelity criterion when the encoder and decoder are allowed to observe side information in the form of a related source $\mathbf{X}_l$.

The following theorems are of our interest:

Theorem 1 *(Superiority of the conditional scheme) Let* $\{X_n\}$ *be a stationary source with memory and let* $R_L^{(\mathbf{X}_k)}$ *be the rate-distortion function for the joint source* $\{\mathbf{X}_k\} = [X_{k-L+1}, ..., X_k]^T$ *with a single-letter distortion measure then*

$$R_L^{(\mathbf{X}_k)}(D) \geq R_L^{(\mathbf{X}_k|\mathbf{X}_l)}(D) \tag{51}$$

with equality iff $\mathbf{X}_l$ *and* $\mathbf{X}_k$ *are independent.*

Proof:

The proof is an extension to blocks of the Theorem 3.1 presented by Gray in [16].□

One further remark is that the theorem above holds not just for a single-letter distortion measure but for compound distortion measures (see discussion [16, pp. 481-482]).

Theorem 2 *(Superiority of the conditional scheme revisited) Let* $\{X_n\}$ *be a stationary source with memory and let* $R_{2L}^{(\mathbf{X}_l\mathbf{X}_k)}$ *be the rate-distortion function for the joint source* $\{\mathbf{X}_l\mathbf{X}_k\} = [X_{k-2L+1}, ..., X_{k-L}, X_{k-L+1}, \ldots$ *with a single-letter distortion measure then*

$$R_{2L}^{(\mathbf{X}_l\mathbf{X}_k)}(D) = \frac{1}{2}R_L^{(\mathbf{X}_l\mathbf{X}_k)}(D) \geq R_L^{(\mathbf{X}_k|\mathbf{X}_l)}(D). \tag{52}$$

with equality iff $\mathbf{X}_l$ *and* $\mathbf{X}_k$ *are independent.*

Proof:

The proof is an extension for blocks of the Theorem 4.1 presented by Gray in [16].□

For optimal coding of the present block, conditioning on the last source block is always superior to optimal coding of the joint block of $2L$ dimensions, since the geometrical constraints of quantization are not operative. High rate conditional vector quantization and conditional rate-distortion theories, although based on different formulations, are consistent.

4 Theoretical Performances

In this section we compare the conditional entropy-constrained VQ performance with the entropy-constrained VQ performance in the high rate region. The definition of the performance gain parallels a similar development by Lookabaugh and Gray [12]. Let us define two measures of conditional entropy advantage (CEA): first, the ratio between per-letter distortions obtained by an entropy-constrained vector quantizer and a conditional entropy-constrained vector quantizer, operating at the same rate and block size L,

$$M_1^c(L, r) = \frac{\hat{D}_L^{\mathrm{E}}}{\hat{D}_L^{\mathrm{CE}}}; \tag{53}$$

and, secondly, the ratio between per-letter distortions of these same quantizers, except that the unconditional ECVQ operates on a block size of $2L$,

$$M_2^c(L,r) = \frac{\hat{D}_{2L}^{\mathrm{E}}}{\hat{D}_L^{\mathrm{CE}}}. \tag{54}$$

Using the assumption of same multiplicative tightness in (38), the latter CEA is the same as the distortion ratio in (40), which is

$$M_2^c(L,r) = G(L/r) \cdot \exp(\frac{r}{2L} I(\mathbf{X}_k; \mathbf{X}_l)). \tag{55}$$

The first CEA is then equal to the ratio of the lower bounds in (32) and (34), where the rates are set equal as in (39), which results in

$$M_1^c(L,r) = \exp(\frac{r}{L} I(\mathbf{X}_k; \mathbf{X}_l)). \tag{56}$$

The CEA quantities, $M_1^c(L,r)$ and $M_2^c(L,r)$ will be used to evaluate the potential gains of conditional entropy-constrained vector quantization over entropy-constrained vector quantization.

Let us consider a well known class of information theoretic sources for our evaluations, the Gaussian autoregressive sources, which are common tractable models for speech and image data. An L-dimensional block $\mathbf{X}$ from a stationary, zero mean, Gaussian sequence with correlation matrix $\mathbf{R}_{XX}$ has a joint entropy given by

$$h(\mathbf{X}) = \frac{1}{2} \log(2\pi e)^L \det(\mathbf{R}_{XX}), \tag{57}$$

where $\det(\mathbf{R}_{XX})$ denotes the determinant of the autocorrelation matrix. Using this fact and the information identity,

$$I(\mathbf{X}_k; \mathbf{X}_l) = 2h(\mathbf{X}_l) - h(\mathbf{X}_k \mathbf{X}_l), \tag{58}$$

the conditional entropy advantages in equations (56) and (55) become

$$\begin{aligned} M_1^c(L,r) &= \left[\frac{\det(\mathbf{R}_{XX}{}^{(L)})}{(\det(\mathbf{R}_{XX}{}^{(2L)}))^{1/2}}\right]^{r/L}, \\ M_2^c(L,r) &= G(L/r)\left[\frac{\det(\mathbf{R}_{XX}{}^{(L)})}{(\det(\mathbf{R}_{XX}{}^{(2L)}))^{1/2}}\right]^{r/2L}. \end{aligned} \tag{59}$$

According to Dembo, Cover and Thomas [17, pp. 1515], the ratio

$$\frac{(\det(\mathbf{R}_{XX}{}^{(L)}))^{1/L}}{(\det(\mathbf{R}_{XX}{}^{(2L)}))^{1/2L}} \geq 1, \tag{60}$$

which is essentially the statement that $I(\mathbf{X}_k;\mathbf{X}_l) \geq 0$. Consequently $M_1^c(L,r) \geq 1$ always and $M_2^c(L,r) \geq 1$, if

$$\frac{(\det(\mathbf{R}_{XX}{}^{(L)}))^{1/L}}{(\det(\mathbf{R}_{XX}{}^{(2L)}))^{1/2L}} \geq (G(L/r))^{-2/r},$$

consistent with (43).

A first-order Gaussian autoregressive (AR(1)) source is generated by

$$X(n) = aX(n-1) + W(n), \tag{61}$$

where a is the correlation coefficient, and $W(n)$ is an i.i.d zero mean Gaussian random sequence with variance σ^2. We will assume that $|a| < 1$, so that this information source is in the stationary regime. It can be shown [18, pp. 116-123] that the correlation sequence is given by

$$R_{XX}(n) = \sigma_X^2 a^{|n|}$$

where σ_X^2 is the variance of the autoregressive source

$$\sigma_X^2 = \frac{\sigma^2}{1-a^2}.$$

A symmetric Toeplitz matrix $\mathbf{R}_{XX} = [R_{XX}(n_2 - n_1)]\ \forall\ n_1, n_2 = 1, ..., L$ is generated by this autocorrelation sequence with determinant given by

$$\det(\mathbf{R}_{XX}{}^{(L)}) = (\sigma_X^2)^L (1-a^2)^{L-1},$$

Evaluating the inter-block mutual information (or memory) in (58) above produces the interesting result,

$$I(\mathbf{X}_k;\mathbf{X}_l) = \log(1-a^2)^{-1/2},$$

which equals the mutual information between the two closest samples from each block. (The per-letter memory decreases as $1/L$.) Consequently the conditional entropy advantages from (59) are found to be

$$\begin{aligned} M_1^c(L,r) &= (1-a^2)^{-r/2L}, \\ M_2^c(L,r) &= G(L/r)(1-a^2)^{-r/4L}. \end{aligned} \tag{62}$$

Note that when we evaluate $\lim_{L\to\infty} M_1^c(L,r)$ and $\lim_{L\to\infty} M_2^c(L,r)$ we obtain the expected unity gains.

In Figure 3 are plotted these CEA's for a Gaussian AR(1) source in dB versus the dimension L, for the case of $r = 2$ (squared error) and $a = 0.9$, which is a reasonable correlation coefficient for modeling tightly correlated data such as speech and images. Note that for small L, substantial advantages are possible using conditional entropy, even when the block size for memoryless VQ is $2L$.

Another common test sequence [19] is the one generated by an AR(2) Gaussian source, defined by

$$X(n) = a_1 X(n-1) + a_2 X(n-2) + W(n) \tag{63}$$

where a's are the regression coefficients and $W(n)$ is the i.i.d zero mean Gaussian random variables with variance σ^2. It can be shown [18, pp. 123-132] that the correlation sequence is given by

$$R_{XX}(n) = \sigma_X^2 \frac{(1-\mu_2^2)\mu_1^{|n|+1} - (1-\mu_1^2)\mu_2^{|n|+1}}{(\mu_1 - \mu_2)(1+\mu_1\mu_2)}$$

where μ_1 and μ_2 are the roots of the quadratic $f(z) = z^2 - a_1 z - a_2$, with $|\mu_1| < 1$ and $|\mu_2| < 1$ required for stability, and where the variance σ_X^2 is equal to

$$\sigma_X^2 = \frac{(1-a_2)\sigma^2}{(1+a_2)(1+a_1-a_2)(1-a_1-a_2)}.$$

In Figure 4 are plotted the conditional entropy advantages for the Gaussian AR(2) source and $r = 2$ with the coefficients $a_1 = 1.515$ and $a_2 = -0.752$. This AR(2) source has often been used to model long-term statistical behavior of speech. Comparing the last two figures reveals that the potential gains for the AR(2) source are higher than those for the AR(1) source for the same L.

5 A new algorithm for conditional entropy-constrained VQ design

In this section, we develop a new algorithm for designing conditional entropy-constrained vector quantizers. We shall assume throughout that the distortion criterion is mean-squared error. The design algorithm developed here will be called conditional entropy-constrained pairwise nearest neighbor design, or in short, CECPNN. Before describing the new algorithm, let us introduce formally a conditional entropy-constrained vector quantizer.

Definition 1 *(first-order conditional block entropy-constrained vector quantization)*

An $L-$dimensional, $M-$level first-order conditional block entropy-constrained quantizer Q, with *associated encoder partition* $\mathcal{E}^{\mathcal{P}} = \{\mathcal{P}_1, ..., \mathcal{P}_M\}$ *and a matrix*

$$\mathbf{S} = \begin{pmatrix} \ell_{1|1} & ,..., & \ell_{1|M} \\ \ell_{2|1} & ,..., & \ell_{2|M} \\ \vdots & \ddots & \vdots \\ \ell_{M|1} & ,..., & \ell_{M|M} \end{pmatrix}$$

of codewords lengths conditioned on the previous codevector indexed by $l = 1, ..., M$*, is a mapping of input vectors* $\mathbf{x}_k \in \Re^L$ *to output vectors* $\mathbf{v} \in \mathcal{C}$ *defined as follows*

$$\mathbf{v} = \mathbf{y}_i \in \mathcal{C},\ \textit{if } d(\mathbf{x}_k, \mathbf{y}_i) + \lambda \ell_{i|l} \leq d(\mathbf{x}_k, \mathbf{y}_m) + \lambda \ell_{m|l};\ \forall i \neq m,\ i, m = 1, ..., M; \tag{64}$$

where $d(.)$ *is the squared error per vector distortion measure between two vectors and the codeword length* $\ell_{k|l}$ *is defined to be the conditional self-entropy, i.e.,*

$$\ell_{k|l} = -\log_2 P_{k|l}.$$

The last detail to be remembered about the implementation of CECVQ is that the encoder performs a non-memoryless mapping, so that an initial state must be given to the encoder and decoder. Throughout this paper we consider that the first vector to be encoded will be quantized by the assignment of the codebook vector that is closest in the distortion sense, i.e., $\lambda = 0$.

The conditional entropy-constrained VQ design, proposed by Chou and Lookabaugh [7], uses the definition above to design these non-memoryless machines using the familiar iteration based on Lloyd's Method I [2] for quantization rule improvement. In what follows, we provide a new alternative for design of these non-memoryless machines.

5.1 CECPNN Design Algorithm

The modified version of the entropy constrained pairwise nearest neighbor algorithm (ECPNN) is introduced here. As a standard reference for this kind of algorithm, we cite Equitz [3]. The ECPNN developed in [6] is an algorithm that designs codebooks by merging the pair of Voronoi regions which gives the least increase of distortion for a given decrease in entropy. The algorithm is suboptimal, because the procedure can not guarantee generation of the best quantizer in the distortion-rate sense. See [6] for a further discussion about this topic. Before describing the CECPNN algorithm, we introduce first some definitions and notation. Let us define a quantizer Q^j, where the superindex j denotes a given step of the algorithm, with a encoder partition given by $\mathcal{E}^{\mathcal{P}j} = \{\mathcal{P}_1^j, ..., \mathcal{P}_{A_j}^j\}$ where A_j is the total number of partitions. Let $\mathcal{C}^j = \{\mathbf{y}_1^j, ..., \mathbf{y}_{A_j}^j\}$ be a codebook where $\mathbf{y}_i^j$'s are the centroids derived from the training set (TS) $\mathcal{T} = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$ which are contiguous L-dimensional sample vectors of the information source $\{X_n\}$. Let the initial codebook size be $A_0 = M$, where a possible initialization for the codebook is the TS, i.e., $\mathbf{y}_i^0 = \mathbf{x}_i^0$ with $A_0 = N = M$.

In order to describe our results in a more convenient way, we will introduce next a notation for an operator named the "merge operator". Let us first introduce the notation $\mathcal{P}^{j+1}(a, b)$ to represent the partition cell which is obtained when we replace, in the set $\mathcal{E}^{\mathcal{P}j}$, the two partition cells $\mathcal{P}_a^j$ and $\mathcal{P}_b^j$ $a, b \in \mathcal{A}^j$ (index set) , $a \neq b$, by their union $\mathcal{P}_a^j \bigcup \mathcal{P}_b^j$. When the merge operation is applied to the A_j-level quantizer Q^j, indicated by $\mathcal{M}_{ab}[Q^j]$, it is understood that a new quantizer $Q^{j+1} = \mathcal{M}_{ab}[Q^j]$ is obtained with $A_j - 1$ levels, that contains the new merged partition cell $\mathcal{P}^{j+1}(a, b)$, is obtained. The quantizer $Q^{j+1}(a, b)$ is said to be the merged (a, b) version of Q^j under operation $\mathcal{M}_{ab}$.

The merging operation results in an increase of distortion per sample ΔD_{ab}^{j+1} and a decrease in rate per sample ΔR_{ab}^{j+1}. The increase of distortion is given by:

$$\Delta D_{ab}^{j+1} = \frac{1}{LN}\{\sum_{\mathbf{x}\in\mathcal{P}_{ab}^j} \parallel \mathbf{x} - \mathbf{y}_{ab}^{j+1} \parallel^2 - \sum_{\mathbf{x}\in\mathcal{P}_a^j} \parallel \mathbf{x} - \mathbf{y}_a^j \parallel^2 - \sum_{\mathbf{x}\in\mathcal{P}_b^j} \parallel \mathbf{x} - \mathbf{y}_b^j \parallel^2\} \tag{65}$$

where $\parallel . \parallel$ is the Euclidean norm of the vector and $\mathbf{y}_{a\ or\ b}^j$ is the centroid of the partition cell $\mathcal{P}_{a\ or\ b}^j$. The vector $\mathbf{y}_{ab}^{j+1}$ denotes the centroid resulted from the merge operator $\mathcal{M}_{ab}$, that is given by:

$$\mathbf{y}_{ab}^{j+1} = \frac{1}{n_a^j + n_b^j}(n_a^j\mathbf{y}_a^j + n_b^j\mathbf{y}_b^j)$$

Manipulating the equation (65), see [3] or [6], the increase of distortion takes the following form:

$$\Delta D_{ab}^{j+1} = \frac{1}{LN}\frac{n_a^j n_b^j}{n_a^j + n_b^j} \parallel \mathbf{y}_a^j - \mathbf{y}_b^j \parallel^2, \tag{66}$$

with n_k^j denoting the un-scaled value of the priori probability mass function P_k^j, that is stored in a vector denoted by $\mathbf{n}^j = [n_k^j]$ for $k = 1, ..., A_0$. Note that a symmetric matrix, $A_0 \times A_0$ could be associated with the increase of distortion and could be used to describe the merging process. Let us define a matrix $\Delta\mathbf{D}^{j+1}$,

$$\Delta\mathbf{D} = \begin{pmatrix} 0 & \Delta D_{12} & \Delta D_{13} & ... & \Delta D_{1A_0} \\ \Delta D_{12} & 0 & \Delta D_{23} & ... & \Delta D_{2A_0} \\ \Delta D_{13} & \Delta D_{23} & 0 & ... & \Delta D_{3A_0} \\ \vdots & \vdots & \ddots & ... & \vdots \\ \Delta D_{1A_0} & \Delta D_{2A_0} & \Delta D_{3A_0} & ... & 0 \end{pmatrix}, \tag{67}$$

where ΔD_{ab} is the increase of distortion as defined before. Note that this symmetric matrix stores all weighted distances, and the null diagonal reflects the fact that $\Delta D_{aa} = 0$. We can incorporate this matrix in our design algorithm in the initialization step and in each iteration, since each time we merge two partition cells, we change the rows a and b and the columns a and b.

When the merge operator is applied on the quantizer Q^j there is also a decrease in block conditional entropy per vector. Initially the conditional entropy associated with Q^j is given by:

$$LR^j = -\sum_{l=1}^{A_j}\sum_{k=1}^{A_j} P^j_{lk}\log_2 P^j_{k|l} = -\sum_{\substack{l=1\\ l\neq\{a,b\}}}^{A_j-2}\sum_{\substack{k=1\\ k\neq\{a,b\}}}^{A_j-2} P^j_{lk}\log_2 P^j_{k|l} \tag{68}$$

$$-\sum_{l=1}^{A_j} P^j_{la}\log_2 P^j_{a|l} - \sum_{l=1}^{A_j} P^j_{lb}\log_2 P^j_{b|l} - \sum_{\substack{k=1\\ k\neq\{a,b\}}}^{A_j-2} P^j_{ak}\log_2 P^j_{k|a} - \sum_{\substack{k=1\\ k\neq\{a,b\}}}^{A_j-2} P^j_{bk}\log_2 P^j_{k|b}$$

and the conditional entropy per vector associated with Q^{j+1} is

$$LR^{j+1} = -\sum_{l=1}^{A_j-1}\sum_{k=1}^{A_j-1} P^{j+1}_{lk}\log_2 P^{j+1}_{k|l} = -\sum_{\substack{l=1\\ l\neq ab}}^{A_j-2}\sum_{\substack{k=1\\ k\neq ab}}^{A_j-2} P^{j+1}_{lk}\log_2 P^{j+1}_{k|l} \tag{69}$$

$$-\sum_{l=1}^{A_j-1} P^{j+1}_{lab}\log_2 P^{j+1}_{ab|l} - \sum_{\substack{k=1\\ k\neq ab}}^{A_j-2} P^{j+1}_{abk}\log_2 P^{j+1}_{k|ab}.$$

Therefore the total decrease of conditional entropy is given by

$$\Delta R^{j+1} = R^j - R^{j+1} = \frac{1}{L}\sum_{k=1}^{A_j-2}\Bigg[\underbrace{P^{j+1}_{abk}\log_2 P^{j+1}_{k|ab}}_{k\neq ab} - \underbrace{P^j_{ak}\log_2 P^j_{k|a}}_{k\neq\{a,b\}} - \underbrace{P^j_{bk}\log_2 P^j_{k|b}}_{k\neq\{a,b\}}\Bigg] \tag{70}$$

$$+\frac{1}{L}\sum_{k=1}^{A_j-1} P^{j+1}_{lab}\log_2 P^{j+1}_{ab|l} - \frac{1}{L}\sum_{k=1}^{A_j}\left[P^j_{la}\log_2 P^j_{a|l} + P^j_{lb}\log_2 P^j_{b|l}\right]$$

In order to simplify the calculation of the drop in entropy given by (70) let us define auxiliary variables. Define the matrix $\mathbf{N}^j$, the so called contingency table, $A_0 \times A_0$ that represents the unscaled value of the joint probability mass function P^j_{lk}, i.e., $\mathbf{N}^j = [n^j_{lk}]$ for k and $l = 1, ..., A_0$. This matrix multiplied by $\frac{1}{N}$ produces the joint probability masses. Note that the unscaled values of the marginal probabilities must follow the relationships:

$$n^j_k = \sum_{l=1}^{A_0} n^j_{lk},$$

$$n^j_l = \sum_{k=1}^{A_0} n^j_{lk},$$

with the index k associated with the unscaled probability value of the vector to be encoded and the index l associated with the unscaled probability value of the previous encoded vector. Of course,

$$n^j_k = n^j_l \;\; \forall\, k = l$$

and, as was mentioned, these values are stored in a vector $\mathbf{n}^j$. We can observe that the rate decrease equation, (70), depends on the matrix $\mathcal{M}_{ab}[\mathbf{N}^j]$. Therefore, for a better understanding, let us consider the merge operator applied to the matrix $\mathbf{N}^j$ in an example. Define a matrix $\mathbf{N}^j$ with size 3×3 and let a and b indicate the possible codevectors to be merged given as follows:

$$\begin{array}{c} \\ a\Rightarrow \\ \\ b\Rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} \overset{a}{\Downarrow} & & \overset{b}{\Downarrow} \end{array} \\ \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \end{array}$$

after a merge operation $\mathcal{M}_{ab}[\mathbf{N}^j]$, the matrix takes the following form,

$$\begin{array}{c} \\ a\Rightarrow \\ \\ b\Rightarrow \end{array} \begin{array}{c} \begin{array}{ccc} \overset{a}{\Downarrow} & & \overset{b}{\Downarrow} \end{array} \\ \begin{pmatrix} A+C+G+I & B+H & 0 \\ D+F & E & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}$$

where the null row and column signifies that the codevector $\mathbf{y}^j_b$ might be deleted from the codebook. The merge operation over the vector $\mathbf{n}^j$ is trivial. We would like to point out, that it is not necessary to evaluate the logarithms for the evaluation of ΔR^{j+1}. A better way is to perform pre-computation of the logarithms from 1 to N and store in a look-up table, and access this table as necessary. The use of this table is one of the keys to fast computation. This kind of operation is not possible for generalized Lloyd type algorithms, such as the CECVQ design algorithm [6].

The average distortion per sample $D^{j+1}(a, b)$ and the average rate per sample $R^{j+1}(a, b)$ obtained when the vectors associated with the TS are quantized with $Q^{j+1}(a, b)$ are then,

$$D^{j+1}(a, b) = D^j + \Delta D_{ab}^{j+1} \tag{71}$$

and

$$R^{j+1}(a, b) = R^j - \Delta R_{ab}^{j+1}. \tag{72}$$

Now we need to decide which pair (a, b) will give the best quantizer in the PNN sense. To discuss optimality, let us consider the quantizer Q^j with distortion D^j and rate R^j. The next quantizer in the sequence is a merged (a, b) version $Q^{j+1}(a, b) = \mathcal{M}_{ab}[Q^j]$, for some value of (a, b) with a rate $R^{j+1}(a, b)$. Let us examine in the distortion versus rate plane two quantizers $\mathcal{M}_{ab}[Q^j]$ and $\mathcal{M}_{a'b'}[Q^j]$ with respect to Q^j. Figure 5 displays the points (R^j, D^j), $(R^{j+1}(a, b), D^{j+1}(a, b))$ and $(R^{j+1}(a', b'), D^{j+1}(a', b'))$ in $R \times D$. We can conclude upon examining this figure that, in the distortion-rate sense, the best sequence of quantizers generated with the merging operation is obtained by selecting the next member of the sequence in such a way that the line joining the corresponding (R, D) points has minimum inclination. The optimality of this least ascendent rule guiding the choice of the next quantizer in the sequence is guaranteed only if the distortion-rate curves are convex [20].

We will denote the ratio of distortion increment to rate decrement obtained in connection with quantizer $Q^{j+1}(a, b)$ by

$$s_{ab}^{j+1} = \frac{\Delta D_{ab}^{j+1}}{\Delta R_{ab}^{j+1}}. \tag{73}$$

The strategy to choose the new quantizer $Q^{j+1}(a, b) = \mathcal{M}_{ab}[Q^j]$ is to search all possible s ratios to find the minimum,

$$s_{\alpha\beta}^{j+1} = \min\ [s_{ab}^{j+1}]\ \forall\ a, b \in \mathcal{A}^j,\ a \neq b, \tag{74}$$

where α and β are the pair of partition cells selected for merging.

The new quantizer generated by the procedure above is formed by the previous and newly merged partition cells and the previous codebook with one deleted codevector.

At this point we can conclude that this procedure is computationally quite intensive when we operate with a large training set. This problem of large computation can be circumvented if we initialize the design procedure with a high rate (measured by the conditional entropy) codebook instead of the training sequence. A reasonable choice for an initial codebook, as adopted by the ECVQ and CECVQ, is one designed by the generalized Lloyd algorithm. With this initialization the initial rate R^0 is equal to H^{GLA} (the first order block conditional entropy), and the initial distortion D^0 is equal D^{GLA}. We are now in position to describe the algorithm.

Algorithm

- Step 1 (Initialization)

  Set $j = 0; A_0 = M; \mathcal{C}^0 = \mathcal{C}^{GLA}; \mathbf{y}_i^0 = \mathbf{y}_i^{GLA}, \mathbf{n}^0 = [n_k]^{GLA}, k = 1, \ldots, M; \mathbf{N}^0 = [n_{lk}]^{GLA}, k = 1, \ldots, M$ and $l = 1, \ldots, M; \mathbf{D}^0 = [\Delta D_{ab}^0]$ $a = 1, \ldots, M$ and $b = 1, \ldots, M; D^0 = D^{GLA}; R^0 = H^{GLA}$;

- Step 2 (Find the best minimum PNN conditional entropy quantizer $Q^{j+1}(a, b)$)

  Get the pair of indices (α, β) for the slope $s_{\alpha\beta}^{j+1}$ such that

$$s_{\alpha\beta}^{j+1} = \min_{a,b \in \mathcal{A}^j, a \neq b} [s_{a,b}^{j+1}]$$

- Step 3 (Quantizer and encoder partition set update)

$$\begin{aligned}
\mathcal{C}^{j+1} &= \mathcal{M}_{\alpha\beta}[\mathcal{C}^j] \\
D^{j+1} &= D^j + \Delta D_{\alpha\beta}^{j+1} \\
R^{j+1} &= R^j - \Delta R_{\alpha\beta}^{j+1} \\
\mathbf{D}^{j+1} &= \mathcal{M}_{\alpha\beta}[\mathbf{D}^j] \\
\mathbf{n}^{j+1} &= \mathcal{M}_{\alpha\beta}[\mathbf{n}^j] \\
\mathbf{N}^{j+1} &= \mathcal{M}_{\alpha\beta}[\mathbf{N}^j]
\end{aligned}$$

- Step 4 (Recursion Counter)

  Set $j = j + 1$

- Step 5 (Stopping Rule)

  If stopping criterion is not met, return to Step 2, else stop.

One choice for the stopping rule at Step 5 is, for instance: stop if $R^{j+1} \leq R^t$ where R^t is a target minimum rate. There are, of course, other choices.

Finally, in order to satisfy the quantization rule in (64) for the data outside the design process we need to define the parameter λ, for each quantizer found by the CECPNN algorithm as

$$\lambda^j = -\frac{D^{j+1} - D^{j-1}}{R^{j+1} - R^{j-1}} \quad \text{with } j = 1, \ldots, \mathcal{J}, \tag{75}$$

where $\mathcal{J} + 1$ is the stopping number of the recursion counter in the algorithm. This parameter has an interpretation as the slope of a line supporting the convex hull of the L-dimensional operational conditional entropy-constrained distortion-rate function.

5.2 Experimental Results

The first result presented in this section pictured in the Figure 6 compares the $D(R)$ distortion-rate function, where D is given in dB, i.e., $D(\text{dB}) = 10\log_{10}(D)$ of an AR(1) and AR(2) process, with source model as given in Section 5, and the theoretical quantizer bound $\hat{D}_L^{CE}$ for $L = 4$, when the fidelity criterion is the mean square error. Instead of using the *vector Shannon lower bound* for $D(R)$, we have used $D(R)$ itself, since it can be evaluated for stationary Gaussian sources [20], from the parametric equations given by:

$$R_\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \max\left[0, \frac{1}{2}\log_2 \frac{S_{XX}(\omega)}{\theta}\right] d\omega, \tag{76}$$

$$D_\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \min\left[\theta, S_{XX}(\omega)\right] d\omega, \tag{77}$$

where $S_{XX}(\omega)$ is the discrete-time source power spectral density (psd) and the nonzero portion of the $R(D)$ (or $D(R)$) curve is generated as the parameter θ traverses the interval $0 \leq \theta \leq$ *ess sup* $S_{XX}(\omega)$, where *ess sup* denotes the essential supremum of a function. The theoretical operational quantizer bound in Figure 6, as expected, lies above the true $D(R)$ at higher rates, where it is valid, and the portion below the $D(R)$ at low rates, where it is invalid, should be disregarded. This Figure is provided mainly for comparison with the operational distortion-rate bound curves given by the implementation of VQ systems presented in this section.

ECPNN, CECPNN and CECVQ codebooks are generated to quantize the output of AR(1) and AR(2) sources with zero mean ($m_X = 0$) and unit variance ($\sigma_X^2 = 1$) subject to mean square error criterion. The distortion is represented in terms of the *signal-to-quantization-noise ratio* (SQNR) measured in dB, defined as $\text{SQNR} = 10\log_{10}(\frac{\sigma_X^2}{D_{mse}})$. To generate the results shown, the sizes of the training sequence and test sequence were 262,144 and 163,840 samples each. All results are outside the training set.

The initial codebook for CECPNN and CECVQ was designed using the GLA with codebook size of 256 codevectors, with dimension $L = 4$. Figure 7 compares the distortion-rate performance between the codebooks generated by CECPNN and CECVQ, when used to quantize the test sequence. We can note that there is indistinguishable difference in performance for an AR(1) source. In the case of the AR(2) source there is an advantage of CECVQ over CECPNN of less than 0.5 dB, when the rate is below 0.5 b/s. In Figure 8 is compared the size of codebooks used for quantization when the codebook is generated by CECVQ and CECPNN. We can see that CECPNN operates with much smaller codebooks. Furthermore, there is a big computational saving using CECPNN codebooks. Here, we want to point out that since we are interested in the distortions obtainable over a whole range of rates, we ran the CECVQ algorithm repeatedly on an increasing sequence of λ's, beginning with $\lambda_0 = 0$ (given by the GLA) and increasing in a manner consistent with obtaining sufficient resolution along the rate axis. The final codebook for λ^j is the initial codebook for λ^{j+1}, as suggested by [5] and [7]. Note that this procedure is an heuristic one to decrease the size of CECVQ codebooks. In order to obtain the codebooks we should operate CECVQ (or ECVQ) with several initial codebook sizes and select the (R, D) points that form the operational convex hull. In practice, since we are generally interested in several rates, this procedure can be very tedious. We should have in mind that CECVQ and ECVQ algorithms exhibit the tendency to populate some partition cells that have small statistical significance. We point out that similar results were obtained when we compared ECPNN and ECVQ for i.i.d Gaussian and AR(1) sources in [6]. The last point, is that CECPNN and ECPNN have the disadvantage that we cannot obtain a continuous distortion-rate curve, only a set of discrete points spaced in a close proximity corresponding to the rate difference of a merge. On the other hand, the CECVQ or ECVQ approach should be able to operate at the exact desired rate through a cut and try procedure to find the appropriate λ.

In Figures 9 and 10, we exhibit the operational distortion-rate curve found by CECPNN and ECPNN for AR(1) and AR(2) sources. The initial codebook size for CECPNN and ECPNN with vector dimension $L = 4$ was again 256 codevectors, and for ECPNN with vector dimension $2L = 8$, it was 2048 codevectors. Also displayed is the quantizer bound $D_{lb,L}^{\mathrm{CE}}$ for $L = 4$. We note that for the AR(1) source, the bound is very close to the operational distortion-rate performance, but, for the AR(2) source, the bound deviates by 2.0 dB. We should remember that the bound is a lower bound derived for high rates, and that our design is not optimal. These figures show that the operational gain of conditional entropy- constrained VQ over entropy-constrained memoryless VQ with the same dimension $L = 4$ is roughly 1.5 dB for the AR(1) source above the rate of 0.3 b/s and 2.0 dB for the AR(2) source above the rate of 0.6 b/s. The CEA, $M_1^c(L, r)$, estimates these gains to be 1.80 and 3.0 dB, respectively, for each source at high rates. Upon doubling the dimension of the entropy constrained VQ to $2L = 8$, these figures show a gain for conditional entropy-constrained VQ with dimension $L = 4$ for the AR(1) source above the rate of 0.6 b/s and for the AR(2) source above the rate of 0.8 b/s. These gains, which reach about 0.2 dB, are notably smaller than those calculated from the corresponding high rate CEA, $M_2^c(L, r)$, which are 0.63 and 1.38 dB, respectively. Evidently, simulations would have to be conducted above the present maximum rate of 1.4 b/s to verify these CEA gains.

Finally, in Figure 11 is plotted the codebook size of the experiments of Figures 9 and 10. We can see that there is a tremendous computational advantage of the conditional entropy-constrained VQ's ($L = 4$) over the entropy-constrained VQ's ($L = 8$). This result is a possible justification for the use of CECVQ's in practice if we are interested in full-search vector quantizers with variable length codewords. Let us further examine the question of complexity for an L-dimensional conditional entropy-constrained vector quantizer when compared with a $2L$-dimensional entropy-constrained vector quantizer. The full search (with look-up table) complexity of L-dimensional VQ (memoryless or non-memoryless) is roughly

$$\begin{gathered} LM \text{ multiplies,} \\ LM \text{ additions and} \\ M - 1 \text{ comparisons.} \end{gathered}$$

With these expressions in mind, let us use the results of Figure 11, in particular for a target rate of $R = 1.0$ b/s. We can see by Table 1 that there is a big computational saving using conditional entropy-constrained vector quantizers. Now, when both schemes work with the same vector dimension, where the distortion-rate performance of the conditional scheme is clearly superior, the computational complexity for the unconditional one is smaller by about a factor of two, as seen in Table II.

The main tradeoff in the use of conditional entropy-constrained vector quantizers is the increase in storage, when both conditional entropy-constrained VQ and entropy-constrained VQ operate with the same vector dimension. However, in certain types of applications the use of these non-memoryless machines may be justified, since they provide excellent distortion-rate performance

Table I

CECPNN of Dimension $L = 4$ versus ECPNN of Dimension $2L = 8$

VQ system	SQNR (dB)	R (b/s)	multiplies	additions	comparisons	source
CE	11.51	0.955	228	228	56	AR(1)
E	11.26	0.974	5824	5824	727	AR(1)
CE	11.88	0.968	268	268	66	AR(2)
E	11.80	0.992	4576	4576	571	AR(2)

Table II

CECPNN of Dimension $L = 4$ versus ECPNN of Dimension $L = 4$

VQ system	SQNR (dB)	R (b/s)	multiplies	additions	comparisons	source
CE	11.51	0.955	228	228	56	AR(1)
E	10.46	0.991	124	124	30	AR(1)
CE	11.88	0.968	268	268	66	AR(2)
E	9.90	0.987	140	140	34	AR(2)

Table III

CECPNN with Dimension $L = 4$ versus ECPNN with Dimension $L = 4$ Storage Requirements

VQ system	SQNR (dB)	R (b/s)	storage	source
CE	11.51	0.955	3477	AR(1)
E	10.46	0.991	155	AR(1)
CE	11.88	0.968	4757	AR(2)
E	9.90	0.987	175	AR(2)

Table IV

CECPNN with Dimension $L = 4$ versus ECPNN with Dimension $2L = 8$ Storage Requirements

VQ system	SQNR (dB)	R (b/s)	storage	source
CE	11.51	0.955	3477	AR(1)
E	11.26	0.974	6552	AR(1)
CE	11.88	0.968	4757	AR(2)
E	11.80	0.992	5148	AR(2)

when compared to the memoryless ones. Roughly, the storage requirements for a conditional entropy-constrained VQ is

$$ML + M^2 \text{ storage units,}$$

and for an entropy-constrained VQ is

$$ML + M \text{ storage units.}$$

Let us compare the storage requirements for a conditional entropy-constrained VQ with $L = 4$ and an entropy-constrained VQ with $L = 4$, for a target bit rate of $R = 1.0$ b/s. The results are shown on Table III. Certainly, in terms of storage the entropy constrained VQ's have a better performance than the conditional entropy-constrained VQ's when both systems operate with the same vector size.

Upon computing the storage requirements for the systems of Table I, we obtain the results shown on Table IV. Note that even in terms of storage, the conditional entropy-constrained VQ's are competitive with the entropy constrained VQ's if the conditional scheme operates with half of the codevector size used by the unconditional one. We would like to point out that the storage requirements depend on the implementation of the entropy coder. For example, Huffman codes do not require storage of probabilities, whereas arithmetic codes do. Consequently our numbers for storage requirements may be inexact, and should be interpreted just as a guideline. Therefore, the last word should be given by the designer, weighing distortion-rate performance, computational complexity and storage.

6 Conclusions

The extension of high-rate quantization theory for conditional entropy-constrained vector quantizers is proposed, as a result a new quantization bound is found, through extension of the works of Gersho [10] and Yamada, Tazaki and Gray [1]. A new design technique for conditional entropy-constrained vector quantization has been suggested. The algorithm, called CECPNN, is based on the same partitional clustering technique used in [6]. The sequence of quantizers recursively generated by the CECPNN algorithm has an operational distortion-rate function that closely approximates the performance of CECVQ, the iterative procedure quantizer design method proposed by Chou and Lookabaugh [7]. This fact has been shown by quantizing autoregressive Gaussian sources. It was verified that the size of the CECPNN codebook is much smaller than the CECVQ codebook when compared at the same distortion-rate performance. We have shown that the performance bound seems to be reasonably tight for the synthetic sources used in this work, especially for the AR(1) source. We have established the main motivation for conditional entropy-constrained VQ's as the reduction of computational complexity, since theoretically and experimentally we have shown that an entropy-constrained VQ must operate with the twice the vector dimension in order to obtain equivalent distortion-rate performance. Future research should be done in the design of coders that use this kind of VQ system, in order to evaluate the possible improvement on the overall distortion-rate performance, computational complexity and storage.

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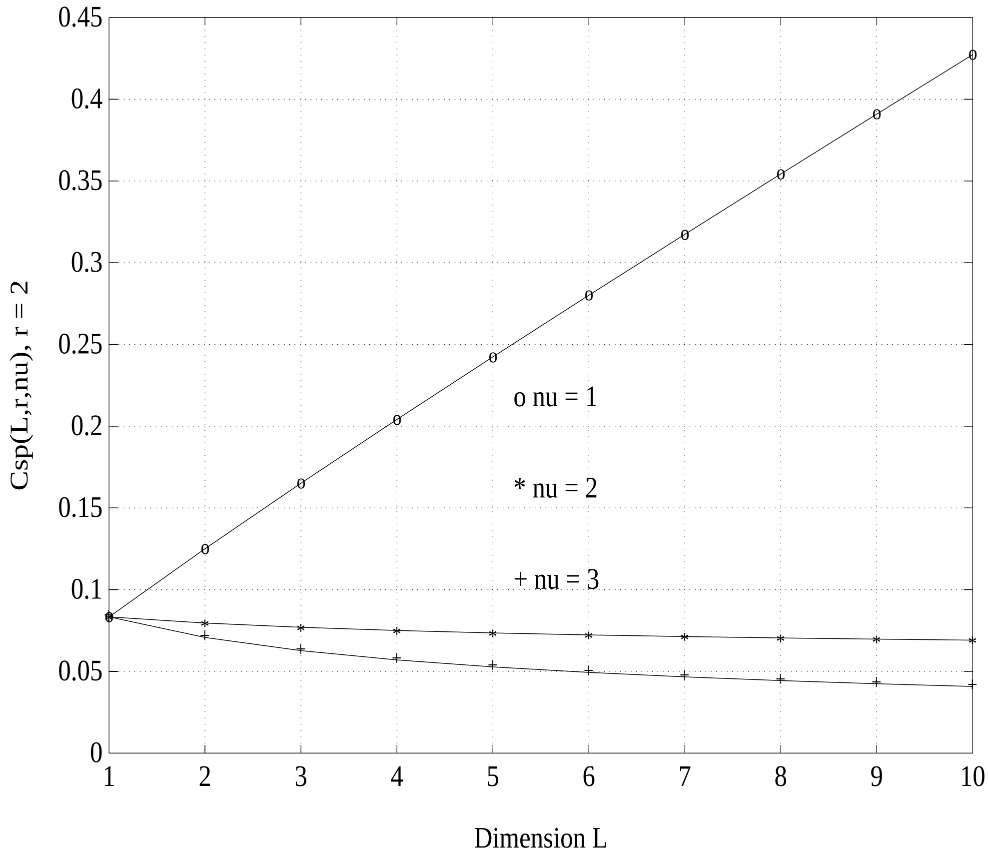

Fig. 1. Spherical quantization coefficient, for $r = 2$
and with different values of ν, as function of L.

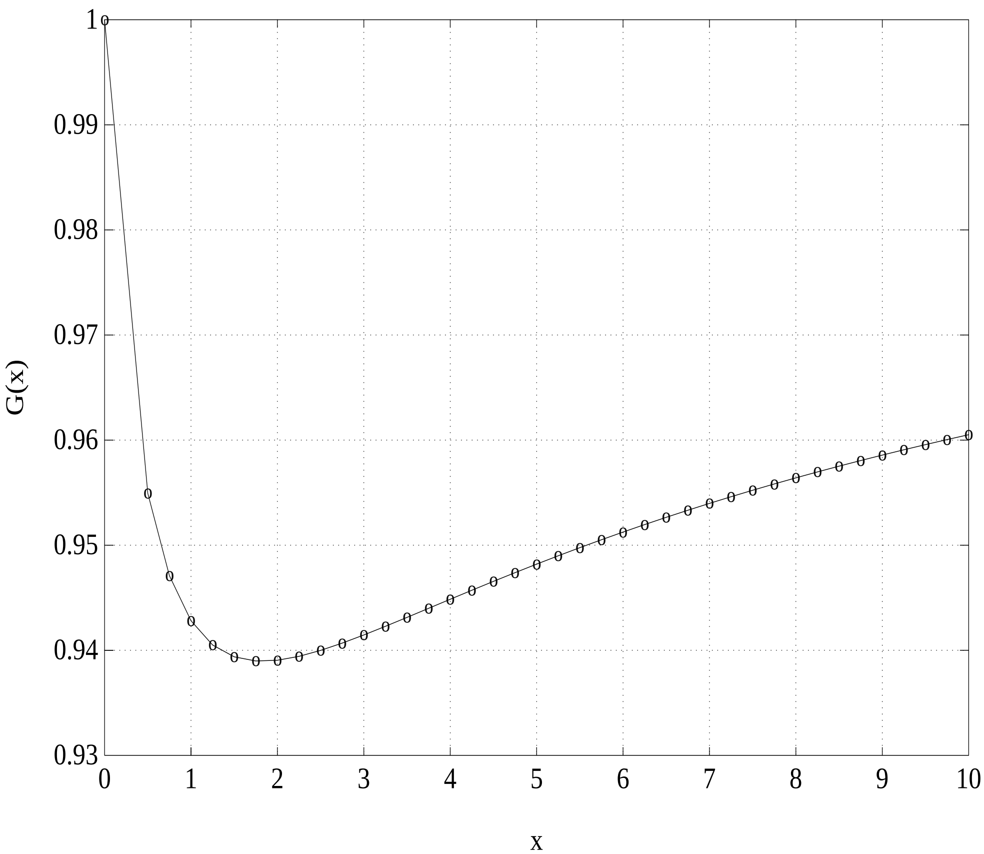

Fig. 2. The function $G(\alpha)$ versus α.

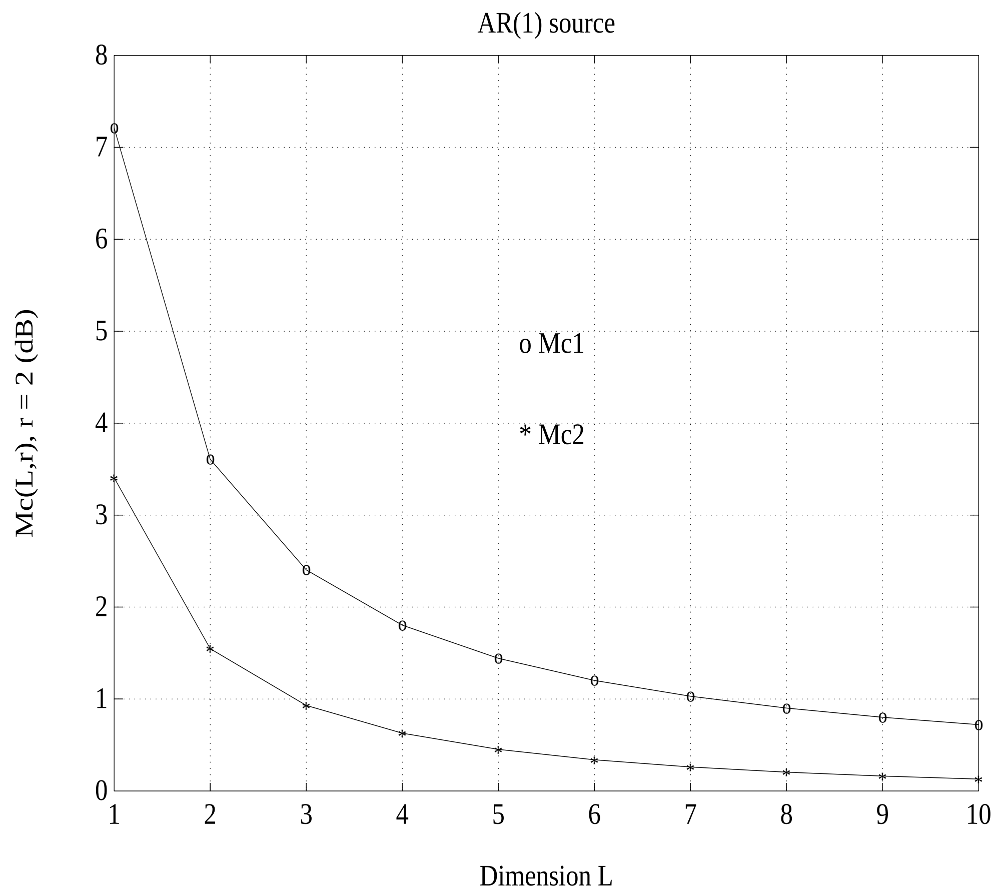

Fig. 3. Conditional entropy advantages in dB versus L,

AR(1), $a = 0.9$, source.

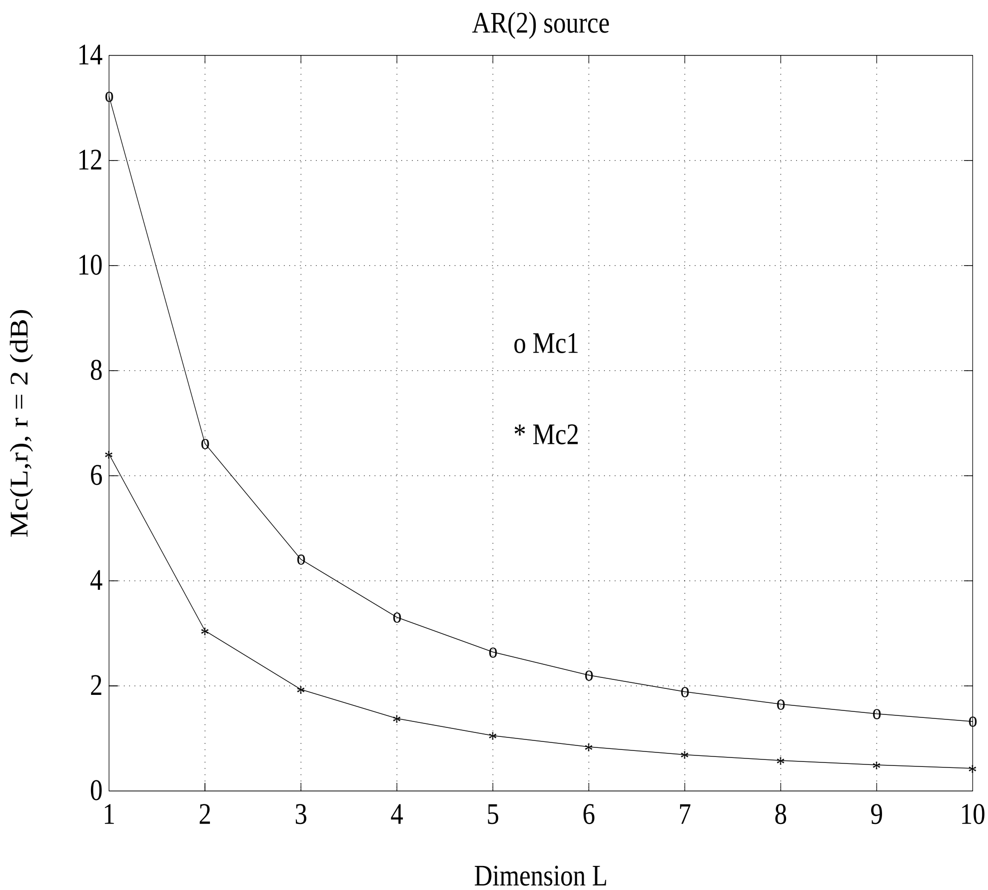

Fig. 4. Conditional entropy advantages in dB versus L,

AR(2), $a_1 = 1.515, a_2 = -0.752$ source.

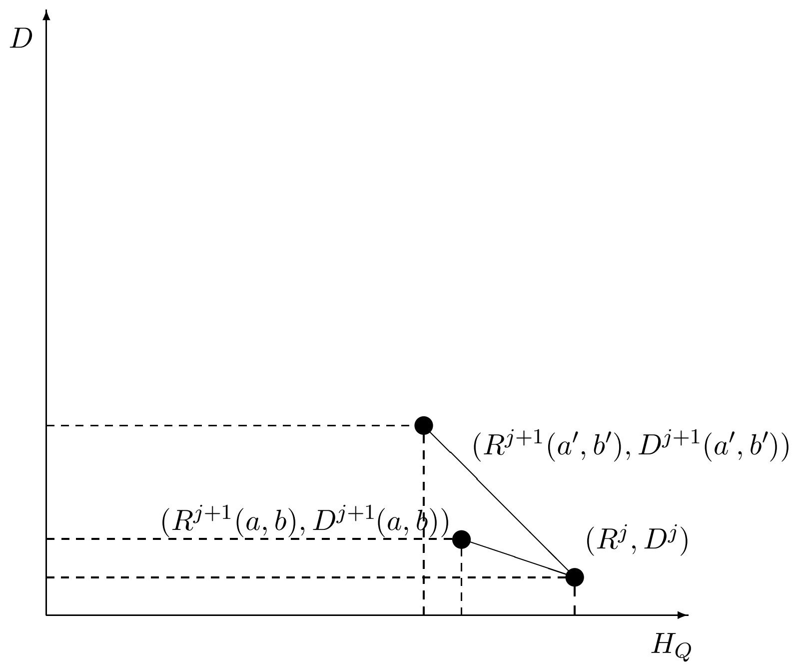

Fig. 5. Distortion versus rate points.

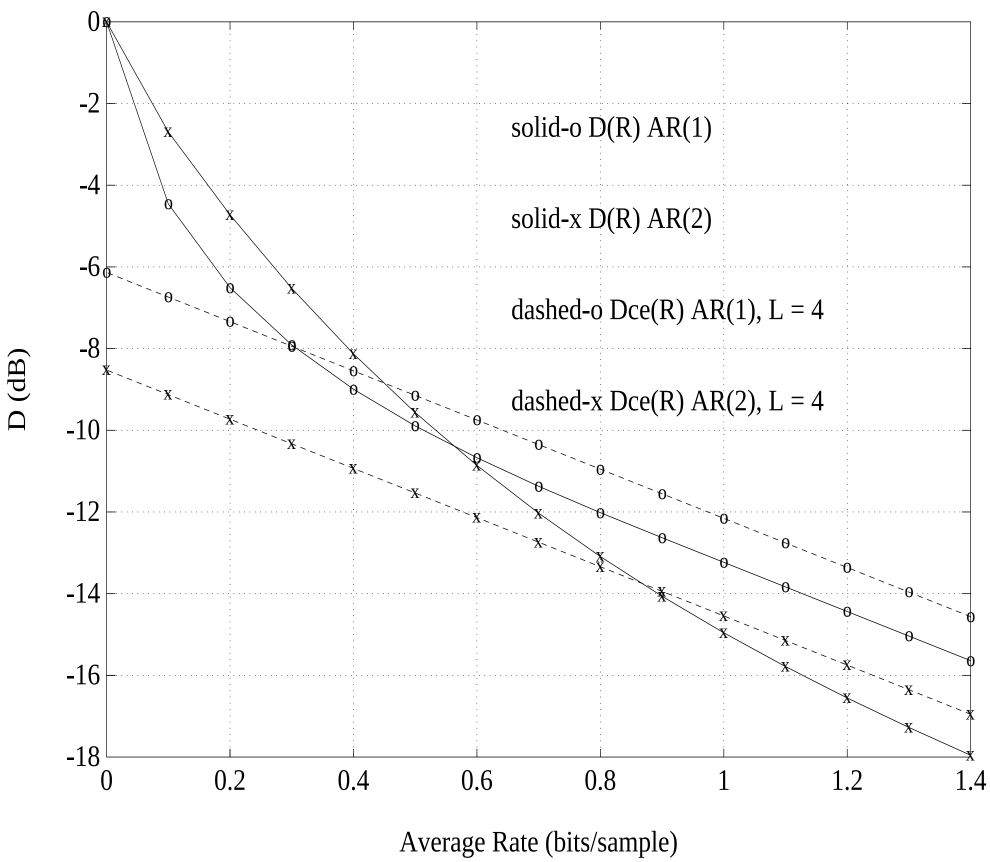

Fig. 6. Comparison between $D(R)$ bound and the $\hat{D}_L^{CE}(R)$ quantization bound, with $L = 4$, for AR(1) and AR(2) sources.

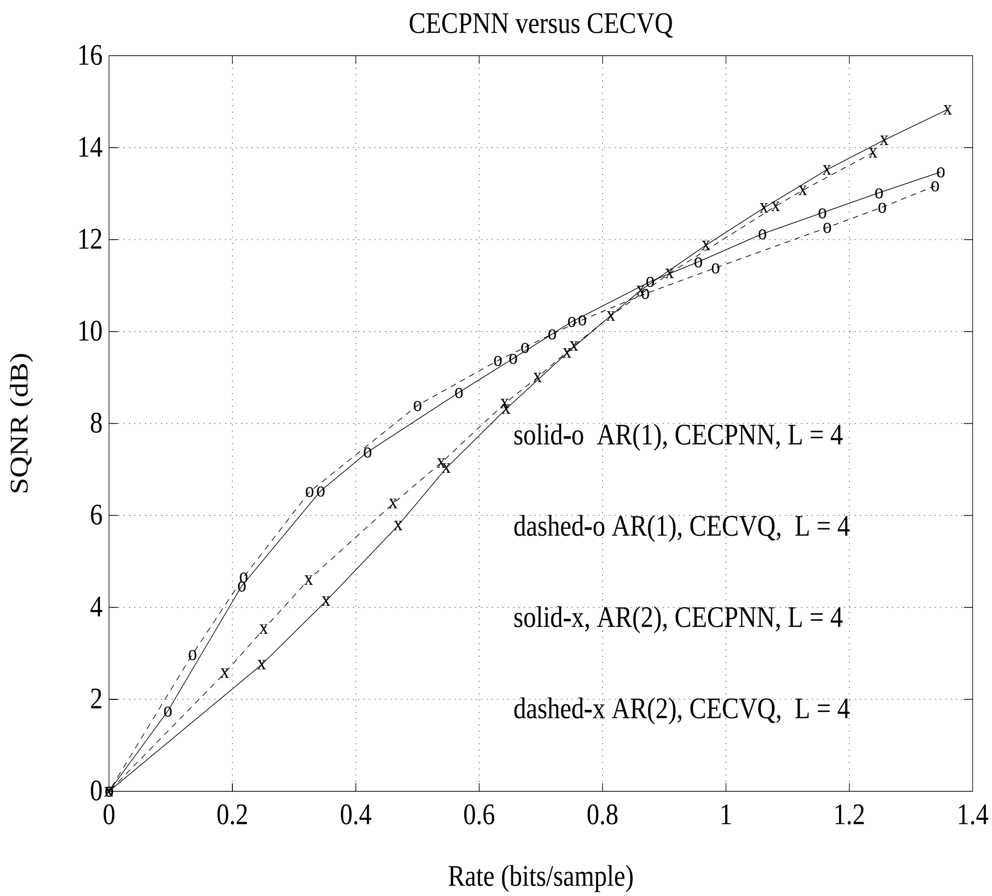

Fig. 7. Distortion-Rate performance for $L = 4$ of CECPNN and CECVQ codebooks for AR(1) and AR(2), outside the TS.

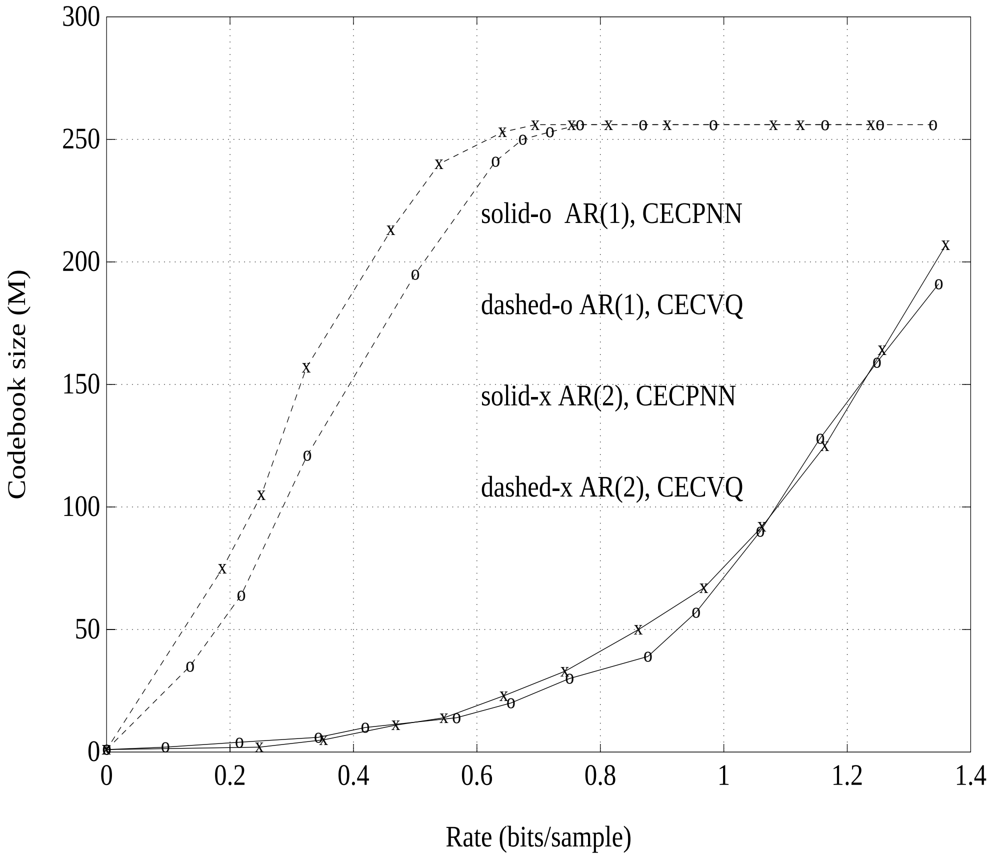

Fig. 8. Size of codebooks designed by CECPNN and CEVQ algorithms, with $L = 4$, for AR(1) and AR(2) sources.

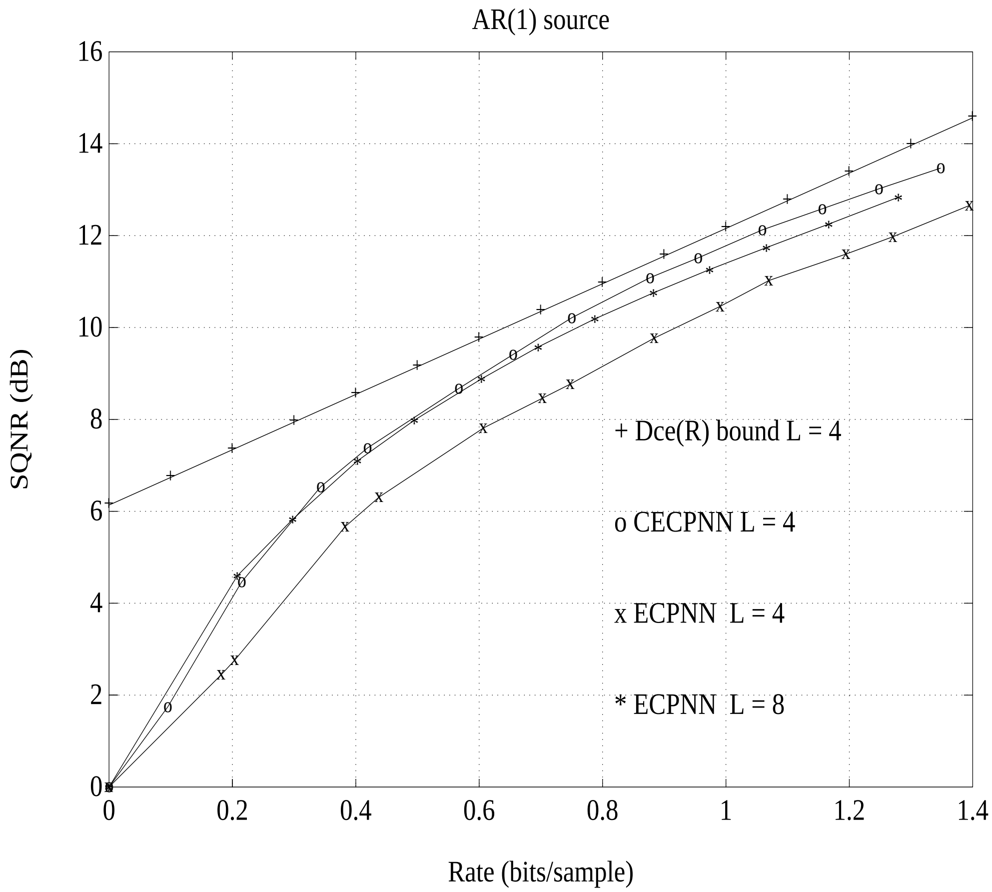

Fig. 9. Distortion-rate performance among codebooks,
designed by CECPNN and ECPNN algorithms,
for AR(1) source, outside the TS.

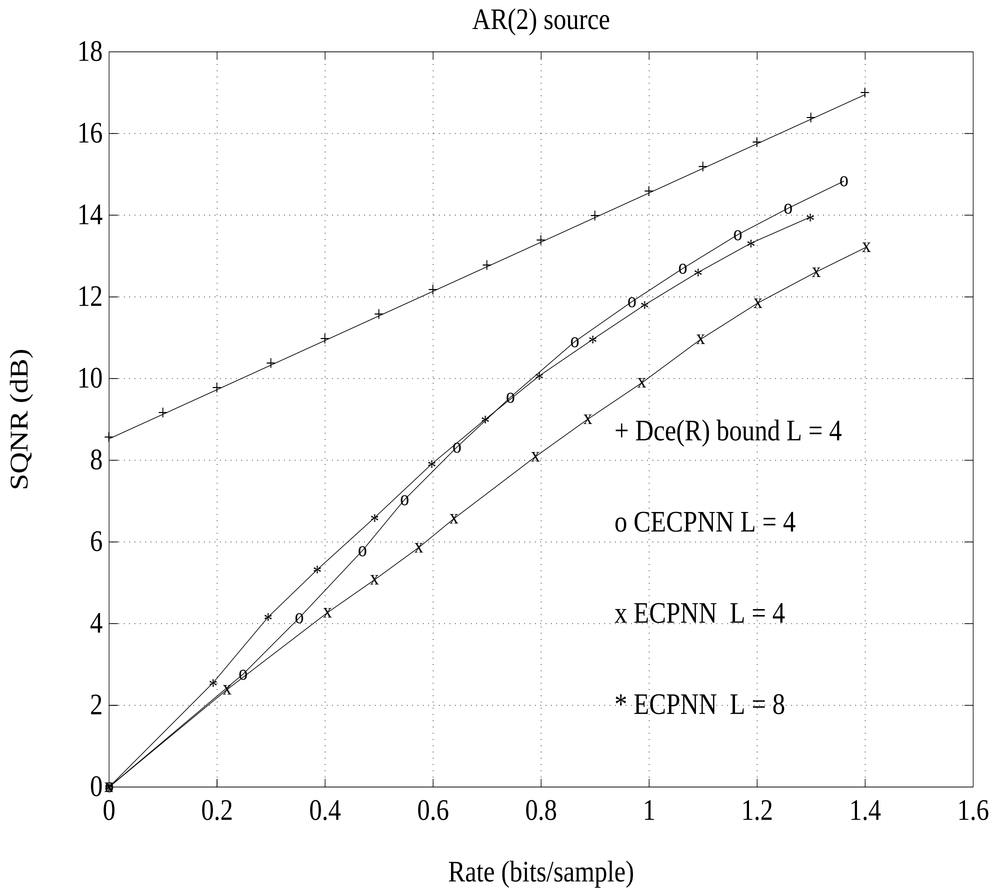

Fig. 10. Distortion-rate performance among codebooks, designed by CECPNN and ECPNN algorithms, for AR(2) source, outside the TS.

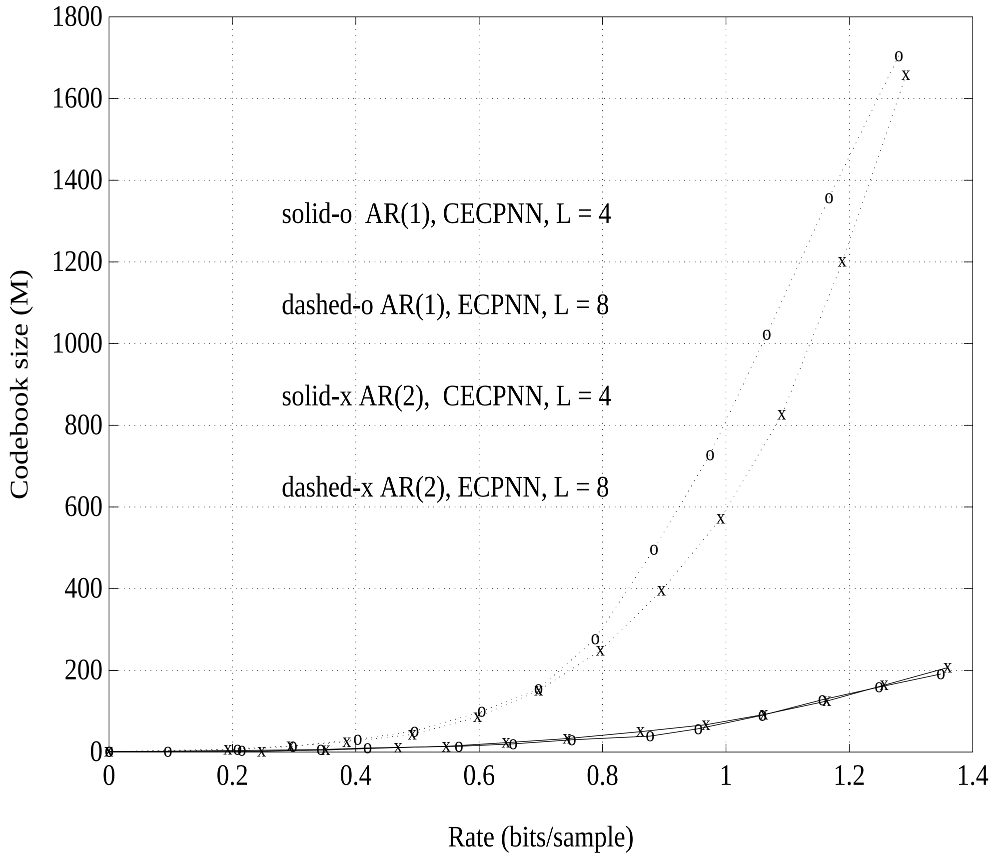

Fig. 11. Size of codebooks designed by CECPNN and ECPNN algorithms, for AR(1) and AR(2) sources.