ECSE 6961: Fundamentals of Wireless Broadband Networks
Homework Problem Set: 4
Due Date: April 1st 2007; [50 points]
Solutions

1. **[Constellations & Orthonormal bases]:** ***[10 pts]***

Show using properties of orthonormal basis functions that if $s_i(t)$ and $s_j(t)$ have constellation points $\mathbf{s}_i$ and $\mathbf{s}_j$, respectively, then

$$||\mathbf{s}_i - \mathbf{s}_j||^2 = \int_0^T (s_i(t) - s_j(t))^2 dt.$$

$s_i(t) = \sum_k s_{ik}\phi_k(t)$
$s_j(t) = \sum_k s_{jk}\phi_k(t)$
where $\{\phi_k(t)\}$ forms an orthonormal basis on the interval [0,T]

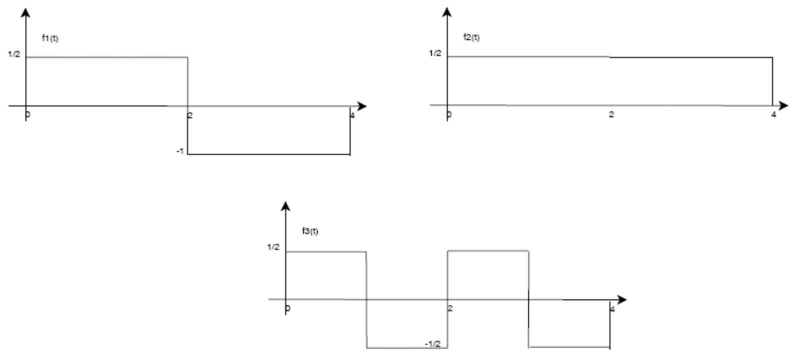
$$\begin{aligned}\int_0^T [s_i(t) - s_j(t)]^2 dt &= \int_0^T \left(\sum_m s_{im}\phi_m(t) - \sum_m s_{jm}\phi_m(t)\right)^2 dt \\ &= \int_0^T \left(\sum_m (s_{im} - s_{jm})\,\phi_m(t)\right)^2 dt\end{aligned}$$

Notice all the cross terms will integrate to 0 due to orthonormal property. So we get

$$\begin{aligned}&= \int_0^T \sum_m (s_{im} - s_{jm})^2\, \phi_m(t)\phi_m(t) dt \\ &= \sum_m (s_{im} - s_{jm})^2 = ||s_i - s_j||^2\end{aligned}$$

2. **[Signal Space]:** ***[10 pts]***

Consider the three signal waveforms $\{\phi_1(t), \phi_2(t), \phi_3(t)\}$ shown below



(a) Show that these waveforms are orthonormal.

(b) Express the waveform $x(t)$ as a linear combination of $\{\phi_i(t)\}$ and find the coefficients, where $x(t)$ is given as

$$x(t) = \begin{cases} -1 & (0 \le t \le 1) \\ 1 & (1 \le t \le 3) \\ 3 & (3 \le t \le 4) \end{cases}$$

(a)

$$< f_1(t), f_2(t) >= \int_0^T f_1(t) f_2(t) dt \quad = \quad 0$$

$$< f_1(t), f_3(t) >= \int_0^T f_1(t) f_3(t) dt \quad = \quad 0$$

$$< f_2(t), f_3(t) >= \int_0^T f_2(t) f_3(t) dt \quad = \quad 0$$

$\Rightarrow f_1(t), f_2(t), f_3(t)$ are orthogonal

Also $\int_0^T f_1(t) f_1(t) = 1$. Similarly for the other waveforms. Thus they are orthonormal.

(b)

$$\begin{aligned}
x(t) &= af_1(t) + bf_2(t) + cf_3(t) \\
0 \le t \le 1 : x(t) &= \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c = -1 \\
1 \le t \le 2 : x(t) &= \frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c = 1 \\
2 \le t \le 3 : x(t) &= -\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c = 1 \\
3 \le t \le 4 : x(t) &= -\frac{1}{2}a + \frac{1}{2}b - \frac{1}{2}c = 3 \\
\Rightarrow a = -2, \quad b = 2, \quad c = -2 & \\
\Rightarrow x(t) &= -2f_1(t) + 2f_2(t) - 2f_3(t)
\end{aligned}$$

3. **[Matched Filter]** [***10 pts***] Find the matched filter for the following waveforms. (Optional: plot them by computer or roughly by hand to compare how it looks w.r.t. the original pulse)

(a) Rectangular pulse: $g(t) = \sqrt{\frac{2}{T}}$

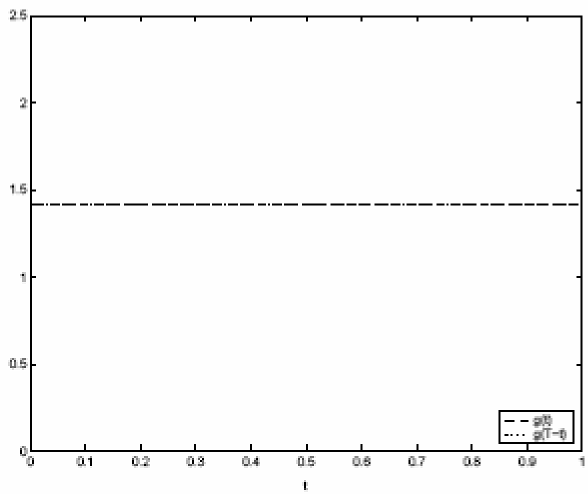
(b) Sinc pulse: $g(t) = \text{sinc}(t)$.

(c) Gaussian pulse: $g(t) = \frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 t^2/\alpha^2}$

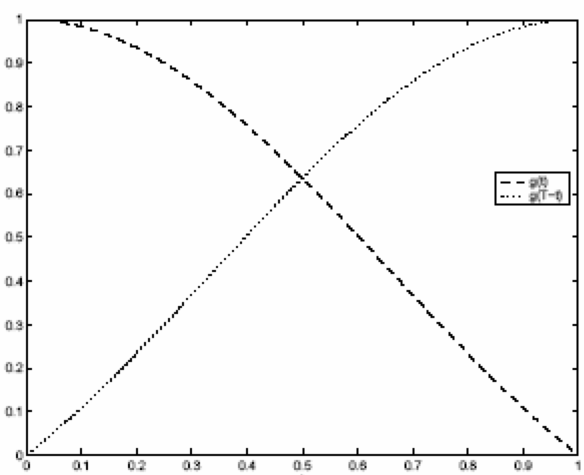
(a) $g(t) = \sqrt{\frac{2}{T}} \quad 0 \le t \le T$

$g(T-t) = \sqrt{\frac{2}{T}} \quad 0 \le t \le T$

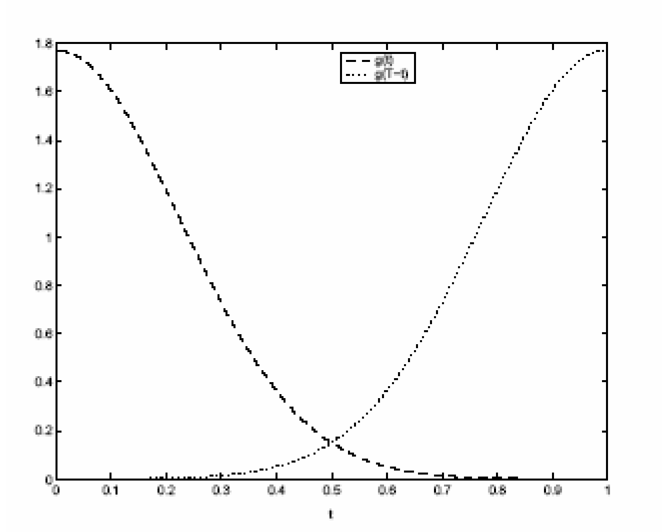
plotted for T=1 , integral value = 2/T = 2



(b) $g(t) = sinc(t) \quad 0 \leq t \leq T$
$g(T-t) = sinc(T-t) \quad 0 \leq t \leq T$
plotted for T=1 , integral value = 0.2470



(c) $g(t) = \frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 t^2/\alpha^2} \quad 0 \leq t \leq T$
$g(T-t) == \frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 (T-t)^2/\alpha^2} \quad 0 \leq t \leq T$
plotted for T=1 , integral value = 0.009



4. [**Modulation perf.**]: [***10 pts***] We saw in class that: $\boxed{P_E(M) \le \frac{1}{M}\sum_{i=1}^{M}\sum_{\substack{k=1 \\ k\neq i}}^{M} P_2(\mathbf{s}_k, \mathbf{s}_i) \le (M-1)Q\left(\frac{d_{\min}/2}{\sqrt{N_0/2}}\right)}$

Assume a target P_E of 10^{-5}. What is the excess required SNR = Es/No (in dB) as we move from BPSK to 64PSK? Explain.
(**Hint**: Recall that in BPSK, the constellation points are at +sqrt(Es) and –sqrt(Es). dmin will reduce. Note: MPSK is a circular constellation. You can also assume the exponential approximation for the Q function if necessary.)

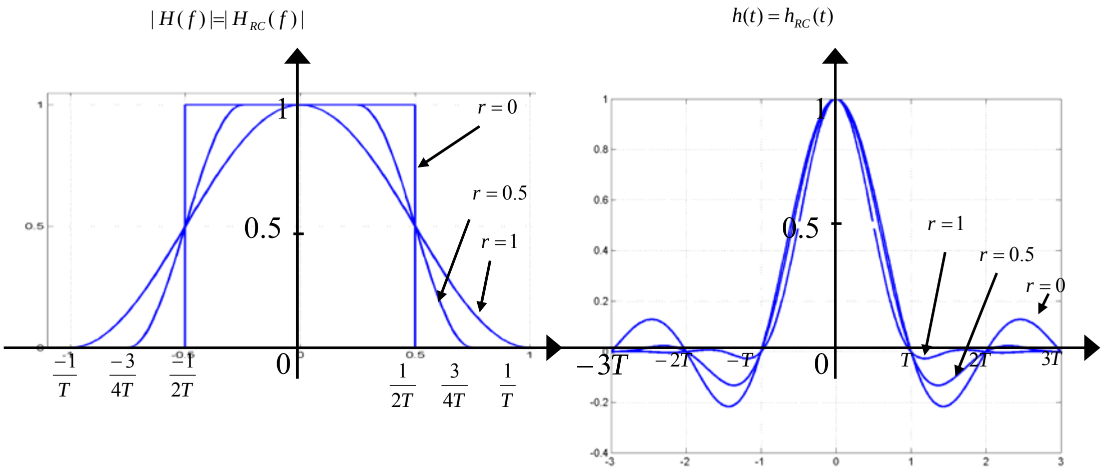
BPSK: M = 2, d_{min} = 2 sqrt(Es), 0.00001 <= Q(sqrt(2 Es/N_0))
Q(x) = 0.5 erfc(x/sqrt(2)) >= 0.00001
- ⇨ x >= erfcinv(0.00002)*sqrt(2) = 4.265
- ⇨ 2 Es/N_0 = x*x = 18.19
- ⇨ Es/N_0 = 9.095 = 9.59 dB

(other approximations possible A. Goldsmith p-167)
64PSK : M = 64, d_{min} = 2 sin(Π/64) sqrt(E's), 0.00001 <= 63 Q(sin(Π/64) sqrt(2 E's/N'$_0$))
- ⇨ Q(sin(Π/64) sqrt(2 E's/N'$_0$)) >= 0.00001/63

  Let Q(y) = 0.5 erfc(y/sqrt(2)) >= 0.00001/63
- ⇨ y >= erfcinv(0.00002/63) *sqrt(2) = 5.1128
- ⇨ $\sin^2$(Π/64) 2 E's/N'$_0$ = y*y = 26.14
- ⇨ E's/N'$_0$ = 26.14/0.0048 = 5445.8 = 37.36 dB

Excess required SNR = 37.36 – 9.59 = 27.77 dB

5. [Pulse Shaping:] ***[10 pts]*** Consider the raised cosine filter (and formula) mentioned in the class slides. Explain the quantitatively the key tradeoffs (vs. the Nyquist Filter) for roll off factors of r = 0, 0.5 and 1.



Tradeoff – The larger the r, the more the bandwidth needed, therefore reducing efficiency. However, the larger the r, the easier the implementation. Also with large r, the flat portion of the frequency response reduces, thus reducing the spreading effect of channel interference.

Excess Bandwidth needed = $W - W_0$ = r W0
Therefore
1) For r = 0, No Excess BW needed
2) For r = 0.5, Excess BW needed = 50%
3) For r = 1, Excess BW needed = 100% (Twice the original BW !)