1. **[Constellations & Orthonormal bases]: [10 pts]**

Show using properties of orthonormal basis functions that if $s_i(t)$ and $s_j(t)$ have constellation points $s_i$ and $s_j$, respectively, then

$$||s_i - s_j||^2 = \int_0^T (s_i(t) - s_j(t))^2 dt$$

$s_i(t) = \sum_k s_{ik} \phi_k(t)$

$s_j(t) = \sum_k s_{jk} \phi_k(t)$

where $\{\phi_k(t)\}$ forms an orthonormal basis on the interval $[0, T]$

$$\int_0^T [s_i(t) - s_j(t)]^2 dt = \int_0^T \left( \sum_m s_{im} \phi_m(t) - \sum_m s_{jm} \phi_m(t) \right)^2 dt$$

$$= \int_0^T \left( \sum_m (s_{im} - s_{jm}) \phi_m(t) \right)^2 dt$$

Notice all the cross terms will integrate to 0 due to orthonormal property. So we get

$$= \int_0^T \sum_m (s_{im} - s_{jm})^2 \phi_m(t) \phi_m(t) dt$$

$$= \sum_m (s_{im} - s_{jm})^2 = ||s_i - s_j||^2$$
2. **[Signal Space]: [10 pts]**

Consider the three signal waveforms \( \{ \phi_1(t), \phi_2(t), \phi_3(t) \} \) shown below.

(a) Show that these waveforms are orthonormal.

(b) Express the waveform \( x(t) \) as a linear combination of \( \{ \phi_i(t) \} \) and find the coefficients, where \( x(t) \) is given as

\[
    x(t) = \begin{cases} 
        -1 & (0 \leq t \leq 1) \\
        1 & (1 \leq t \leq 3) \\
        3 & (3 \leq t \leq 4) 
    \end{cases}
\]

(a)

\[
    \langle f_1(t), f_2(t) \rangle = \int_0^T f_1(t)f_2(t)dt = 0 \\
    \langle f_1(t), f_3(t) \rangle = \int_0^T f_1(t)f_3(t)dt = 0 \\
    \langle f_2(t), f_3(t) \rangle = \int_0^T f_2(t)f_3(t)dt = 0
\]

\[\Rightarrow f_1(t), f_2(t), f_3(t) \text{ are orthogonal}\]

Also \( \int_0^T f_1(t)f_1(t) = 1 \). Similarly for the other waveforms. Thus they are orthonormal.
3. [Matched Filter] [10 pts] Find the matched filter for the following waveforms. (Optional: plot them by computer or roughly by hand to compare how it looks w.r.t. the original pulse)

(a) Rectangular pulse: \( g(t) = \frac{2}{T} \)

(b) Sinc pulse: \( g(t) = \text{sinc}(t) \)

(c) Gaussian pulse: \( g(t) = \frac{\sqrt{\pi}}{\sigma^2} e^{-\pi^2 t^2 / \sigma^2} \)

(a) \( g(t) = \sqrt{\frac{2}{T}} \quad 0 \leq t \leq T \)
\[ g(T - t) = \sqrt{\frac{2}{T}} \quad 0 \leq t \leq T \]
plotted for \( T = 1 \), integral value = \( 2/T = 2 \)
(b) \( g(t) = \text{sinc}(t) \quad 0 \leq t \leq T \)
\[ g(T - t) = \text{sinc}(T - t) \quad 0 \leq t \leq T \]
plotted for \( T=1 \), integral value = 0.2470

(c) \( g(t) = \frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 t^2 / \alpha^2} \quad 0 \leq t \leq T \)
\[ g(T - t) = \frac{\sqrt{\pi}}{\alpha} e^{-\pi^2 (T-t)^2 / \alpha^2} \quad 0 \leq t \leq T \]
plotted for \( T=1 \), integral value = 0.009
4. [Modulation perf.]: [10 pts] We saw in class that:

\[ P_e(M) \leq \frac{1}{M} \sum_{i=1}^{M} \sum_{k=1}^{M} P_s(s_i, s_k) \leq (M-1) Q\left( \frac{d_{min}}{2\sqrt{N_0}} \right) \]

Assume a target \( P_e \) of \( 10^{-5} \). What is the excess required SNR = \( E_s/No \) (in dB) as we move from BPSK to 64PSK? Explain.

(Hint: Recall that in BPSK, the constellation points are at +sqrt(\( E_s \)) and −sqrt(\( E_s \)). \( d_{min} \) will reduce. Note: MPSK is a circular constellation. You can also assume the exponential approximation for the Q function if necessary.)

BPSK: \( M = 2 \), \( d_{min} = 2 \sqrt{E_s} \), \( 0.00001 \leq Q(\sqrt{2 E_s/N_0}) \)
\[
Q(x) = 0.5 \text{erfc}(x/\sqrt{2}) \geq 0.00001
\]
\[
\Rightarrow x \geq \text{erfcinv}(0.00002) \times \sqrt{2} = 4.265
\]
\[
\Rightarrow 2 E_s/N_0 = x^2 = 18.19
\]
\[
\Rightarrow E_s/N_0 = 9.095 = 9.59 \text{ dB}
\]

(Other approximations possible A. Goldsmith p-167)

64PSK: \( M = 64 \), \( d_{min} = 2 \sin(\Pi/64) \sqrt{E_s} \), \( 0.00001 \leq 63 Q(\sin(\Pi/64) \sqrt{2 E_s/N_0'}) \)
\[
Q(\sin(\Pi/64) \sqrt{2 E_s/N_0'}) \geq 0.00001/63
\]
\[
\Rightarrow y \geq \text{erfcinv}(0.00002/63) \times \sqrt{2} = 5.1128
\]
\[
\Rightarrow \sin^2(\Pi/64) 2 E_s/N_0' = y^2 = 26.14
\]
\[
\Rightarrow E_s/N_0' = 26.14/0.0048 = 5445.8 = 37.36 \text{ dB}
\]

Excess required SNR = 37.36 − 9.59 = 27.77 dB
5. [Pulse Shaping:] [10 pts] Consider the raised cosine filter (and formula) mentioned in the class slides. Explain the quantitatively the key tradeoffs (vs. the Nyquist Filter) for roll off factors of \( r = 0, 0.5 \) and 1.

**Tradeoff** – The larger the \( r \), the more the bandwidth needed, therefore reducing efficiency. However, the larger the \( r \), the easier the implementation. Also with large \( r \), the flat portion of the frequency response reduces, thus reducing the spreading effect of channel interference.

Excess Bandwidth needed = \( W - W_0 = r W_0 \)

Therefore

1) For \( r = 0 \), No Excess BW needed
2) For \( r = 0.5 \), Excess BW needed = 50%
3) For \( r = 1 \), Excess BW needed = 100% (Twice the original BW !)