WBN : Homework #2
Probability and Stochastic Processes
[40 points ,Due 20th Feb 2007 in class]

1. **[6 pts]** A fair coin is tossed repeatedly until the first head appears.

(i) **[2 pts]** Find the probability that the first head appears on the k^{th} toss. Let us call this event E_k.

(ii) **[2 pts]** Let $S = \bigcup_{i=1}^{\infty} E_i$. Verify that $P(S) = 1$.

(iii) **[2 pts]** Show that the union bound is tight for the event that first head appears in any of the first t tosses, i.e., the probability of the above event equals $\Sigma_{i=1..t} P(E_i)$.

2. **[4 pts]** For a continuous Random Variable $\mathbf{X}$, and $a > 0$, show that (Chebyshev inequality) : $P(|\mathbf{X} - \mu_{\mathbf{x}}| >= a) <= \sigma_{\mathbf{x}}^2/a^2$. <u>Hint</u>: use the definition of variance ($\sigma_{\mathbf{x}}^2 = \int(\mathbf{X} - \mu_{\mathbf{x}})^2 f(\mathbf{X})d\mathbf{X}$). (note: there are several ways to derive this inequality. Any way you do it is fine.)

3. **[5 pts]** Let **X** be a uniform random variable over (-1, 1). Let **Y** = $\mathbf{X}^n$.

(i) **[3 pts]** Calculate the covariance of **X** and **Y, i.e. E[XY] – E[X]E[Y].**

(ii) **[2 pts]** Calculate the correlation coefficient of **X** and **Y**.

4. **[5 pts]** A laboratory test to detect a certain disease has the following statistics. Let

X = event that the tested person has the disease

Y = event that the test result is positive

It is known that 0.1 percent of the population actually has the disease. Also, P(**Y** | **X**) = 0.99 and P(**Y** | $\mathbf{X}^{\mathbf{c}}$) = 0.005. What is the probability that a person has the disease given that the test result is positive ?

5. **[5 pts]** Let $(X_1, \ldots, X_n)$ be a random sample of an exponential random variable **X** with unknown parameter λ. Determine the maximum-likelihood estimator of λ. Hint: look at slides 40, 73 to review the exponential r.v. and MaxLikelihood estimator.

6. **[15 pts]** Consider the random process $Y(t) = (-1)^{X(t)}$, where X(t) is a Poisson process with rate λ. Thus Y(t) starts at Y(0) = 1 and switches back and forth from +1 to -1 at random Poisson times T_i. Note: P(Y(t) b= 1) = exp(- λt) cos h λt; P(Y(t) = -1) = exp(- λt) sin h λt, because these are the sum of even or odd terms of a poisson distribution.

(i) **[3 pts]** Find the mean of Y(t).

(ii) **[3 pts]** Find the autocorrelation function of Y(t).

(iii) **[6 pts]** Let Z(t) = A Y(t) where A is a discrete random variable independent of Y(t) and takes on values 1 and -1 with equal probability. Show that Z(t) is WSS. (note: Y(t) is not WSS)

(iv) **[3 pts]** Find the power spectral density of Z(t). Note: look up the fourier transform table w/ autocorrelation function of Z(t).