WBN : Homework #2
Probability and Stochastic Processes
[40 points ,Due 20th Feb 2007 in class]

1. [6 pts] A fair coin is tossed repeatedly until the first head appears.
   (i) [2 pts] Find the probability that the first head appears on the kth toss. Let us call this event \( E_k \).
   (ii) [2 pts] Let \( S = \bigcup_{i=1}^{\infty} E_i \). Verify that \( P(S) = 1 \).
   (iii) [2 pts] Show that the union bound is tight for the event that first head appears in any of the first \( t \) tosses, i.e., the probability of the above event equals \( \sum_{i=1}^{t} P(E_i) \).

2. [4 pts] For a continuous Random Variable \( X \), and \( a > 0 \), show that (Chebyshev inequality): \( P(|X - \mu_X| \geq a) \leq \frac{\sigma_X^2}{a^2} \). Hint: use the definition of variance \( \sigma_X^2 = \int (X - \mu_X)^2 f(X) dX \). (note: there are several ways to derive this inequality. Any way you do it is fine.)
3. **[5 pts]** Let $X$ be a uniform random variable over $(-1, 1)$. Let $Y = X^n$.

(i) **[3 pts]** Calculate the covariance of $X$ and $Y$, i.e. $E[XY] - E[X]E[Y]$.

(ii) **[2 pts]** Calculate the correlation coefficient of $X$ and $Y$.

4. **[5 pts]** A laboratory test to detect a certain disease has the following statistics. Let

$X = \text{event that the tested person has the disease}$

$Y = \text{event that the test result is positive}$

It is known that 0.1 percent of the population actually has the disease. Also, $P(Y | X) = 0.99$ and $P(Y | X^c) = 0.005$. What is the probability that a person has the disease given that the test result is positive?
5. **[5 pts]** Let \((X_1, \ldots, X_n)\) be a random sample of an exponential random variable \(X\) with unknown parameter \(\lambda\). Determine the maximum-likelihood estimator of \(\lambda\). **Hint:** look at slides 40, 73 to review the exponential r.v. and MaxLikelihood estimator.

6. **[15 pts]** Consider the random process \(Y(t) = (-1)^{X(t)}\), where \(X(t)\) is a Poisson process with rate \(\lambda\). Thus \(Y(t)\) starts at \(Y(0) = 1\) and switches back and forth from +1 to -1 at random Poisson times \(T_i\). **Note:** \(P(Y(t) = 1) = \exp(-\lambda t) \cos \lambda t; P(Y(t) = -1) = \exp(-\lambda t) \sin \lambda t\), because these are the sum of even or odd terms of a poisson distribution.

(i) **[3 pts]** Find the mean of \(Y(t)\).

(ii) **[3 pts]** Find the autocorrelation function of \(Y(t)\).

(iii) **[6 pts]** Let \(Z(t) = A \cdot Y(t)\) where \(A\) is a discrete random variable independent of \(Y(t)\) and takes on values 1 and -1 with equal probability. Show that \(Z(t)\) is WSS. (note: \(Y(t)\) is not WSS)

(iv) **[3 pts]** Find the power spectral density of \(Z(t)\). **Note:** look up the fourier transform table w/ autocorrelation function of \(Z(t)\).