WBN : Homework #2 - SOLUTIONS
Probability and Stochastic Processes

1. A fair coin is tossed repeatedly until the first head appears.

(i) Find the probability that the first head appears on the kth toss. Let us call this event E_k. ($\frac{1}{2^k}$)

(ii) Let $S = \bigcup_{i=1}^{\infty} E_i$. Verify that P(S) = 1. ($\sum_{i=1}^{\infty} \frac{1}{2^i} = 1$)

(iii) Show that the union bound is tight for the event that first head appears in any of the first t tosses, i.e., the probability of the above event equals $\Sigma_{i=1..t} P(E_i)$.

(Let A_t denote the event that the first head appears in any of the first t tosses. $P(A_t) = 1 - P(\text{first t tosses are all tails}) = 1 - \frac{1}{2^t}$. Now, $\Sigma_{i=1..t} P(E_i) = \sum_{i=1}^{t} \frac{1}{2^i}$

$1 - \frac{1}{2^t} = P(A_t)$)

2. For a continuous Random Variable **X**, and a > 0, show that (Chebyshev inequality) : $P(|\mathbf{X} - \mu_x| >= a) <= \sigma_x^2/a^2$.

($\sigma_x^2 = \int_{-\infty}^{\infty}(x-u_x)^2 f(x)dx \geq \int_{|x-u_x|\geq a}(x-u_x)^2 f(x)dx \geq a^2 \int_{|x-u_x|\geq a} f(x)dx = a^2 P(|x-u_x| \geq a)$)

3. Let **X** be a uniform random variable over (-1, 1). Let $\mathbf{Y} = \mathbf{X}^n$.

(i) Calculate the covariance of **X** and **Y**. (E[X] = 0; cov (X,Y) = E[XY] – E[X]E[Y] = E[XY] = $E[X^{n+1}]$ = 1/(n+2) if n is odd; 0 if n is even.)

(ii) Calculate the correlation coefficient of **X** and **Y**. ($\sigma_x = 1/\sqrt{3}$; $\sigma_Y = 1/\sqrt{(2n+1)}$; cor(X, Y) = cov(X,Y)/($\sigma_x \sigma_Y$) = $\sqrt{3(2n+1)}/(n+2)$ if n is odd; 0 if n is even.)

4. A laboratory test to detect a certain disease has the following statistics. Let

X = event that the tested person has the disease

Y = event that the test result is positive

It is known that 0.1 percent of the population actually has the disease. Also, $P(\mathbf{Y} \mid \mathbf{X}) = 0.99$ and $P(\mathbf{Y} \mid \mathbf{X}^c) = 0.005$. What is the probability that a person has the disease given that the test result is positive ?

$P(X|Y) = P(Y|X)P(X)/P(Y)$ where $P(Y) = P(Y|X)P(X) + P(Y|X^c)P(X^c)$. Therefore

$P(X|Y) = (0.99)(0.001)/[(0.99)(0.001) + (0.005)(0.999)] = 0.1654$.

Note that in only 16.5% of the cases where the tests are positive will the person actually have the disease even though the test is 99% effective in detecting the disease when it is, in fact, present.

5. Let $(X_1, \ldots, X_n)$ be a random sample of an exponential random variable **X** with unknown parameter λ. Determine the maximum-likelihood estimator of λ.

$L(\lambda) = P(X_1, \ldots, X_n \mid \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda n \overline{X}}$

$\log L(\lambda) = n \log \lambda - \lambda n \overline{X}$

Equating $d/d\lambda\, [\log L(\lambda)] = 0$, we get MLE $\lambda = 1 / \overline{X}$

6. Consider the random process $Y(t) = (-1)^{X(t)}$, where $X(t)$ is a Poisson process with rate λ. Thus $Y(t)$ starts at $Y(0) = 1$ and switches back and forth from +1 to -1 at random Poisson times T_i.

(i) Find the mean of $Y(t)$. $Y(t) = 1$ if $X(t)$ is even; -1 if $X(t)$ is odd.

$P(Y(t) = 1) = \exp(-\lambda t) \cos h\, \lambda t$; $P(Y(t) = -1) = \exp(-\lambda t) \sin h\, \lambda t$.

Hence, $E[Y(t)] = \exp(-\lambda t) (\cos h\, \lambda t - \sin h\, \lambda t) = \exp(-2\lambda t)$.

(ii) Find the autocorrelation function of $Y(t)$.

$Y(t)Y(t+\tau) = 1$ if there are an even number of events in $(t, t+\tau)$; -1 otherwise.

$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)] = \exp(-2\lambda\tau)$. Thus, $R_Y(\tau) = \exp(-2\lambda|\tau|)$.

(iii) Let $Z(t) = A\, Y(t)$ where A is a discrete random variable independent of $Y(t)$ and takes on values 1 and -1 with equal probability. Show that $Z(t)$ is WSS.

$E[A] = 0$; $E[A^2] = 1$; $E[Z(t)] = E[A]E[Y(t)] = 0$; $R_Z(\tau) = R_Y(\tau) = \exp(-2\lambda|\tau|)$.

(iv) Find the power spectral density of $Z(t)$.

$4\lambda / (\omega^2 + 4\lambda^2)$