1. A fair coin is tossed repeatedly until the first head appears.

(i) Find the probability that the first head appears on the k\textsuperscript{th} toss. Let us call this event $E_k$. \left( \frac{1}{2^k} \right)

(ii) Let $S = \bigcup_{i=1}^{\infty} E_i$. Verify that $P(S) = 1$. \left( \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 \right)

(iii) Show that the union bound is tight for the event that first head appears in any of the first t tosses, i.e., the probability of the above event equals $\sum_{i=1..t} P(E_i)$.

(Let $A_t$ denote the event that the first head appears in any of the first t tosses. $P(A_t) = 1 - P(\text{first t tosses are all tails}) = 1 - \frac{1}{2^t}$. Now, $\sum_{i=1}^{t} \frac{1}{2^i} = 1 - \frac{1}{2^t} = P(A_t)$)
2. For a continuous Random Variable $X$, and $a > 0$, show that (Chebyshev inequality): $P(|X - \mu| \geq a) \leq \frac{\sigma_x^2}{a^2}$.

\[
\sigma_x^2 = \int_{-\infty}^{\infty} (x - u_x)^2 f(x)dx \geq \int_{|x-u_x| \geq a} (x - u_x)^2 f(x)dx \geq a^2 \int_{|x-u_x| \geq a} f(x)dx = a^2 P(|x-u_x| \geq a)
\]

3. Let $X$ be a uniform random variable over $(-1, 1)$. Let $Y = X^n$.

(i) Calculate the covariance of $X$ and $Y$. ($E[X] = 0$; $\text{cov} (X,Y) = E[XY] - E[X]E[Y] = E[XY] = E[X^{n+1}] = 1/(n+2)$ if $n$ is odd; 0 if $n$ is even.)

(ii) Calculate the correlation coefficient of $X$ and $Y$. ($\sigma_x = 1/\sqrt{3}$; $\sigma_Y = 1/\sqrt{(2n+1)}$; $\text{cor}(X, Y) = \text{cov}(X,Y)/(\sigma_x \sigma_Y) = \sqrt{3(2n+1)/(n+2)}$ if $n$ is odd; 0 if $n$ is even.)
4. A laboratory test to detect a certain disease has the following statistics. Let

\( X \) = event that the tested person has the disease \\
\( Y \) = event that the test result is positive \\

It is known that 0.1 percent of the population actually has the disease. Also, \( P(Y \mid X) = 0.99 \) and \( P(Y \mid X^c) = 0.005 \). What is the probability that a person has the disease given that the test result is positive?

\[
P(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)} = \frac{P(Y \mid X)P(X)}{P(Y \mid X)P(X) + P(Y \mid X^c)P(X^c)}.
\]

Therefore

\[
P(X \mid Y) = \frac{(0.99)(0.001)}{[(0.99)(0.001) + (0.005)(0.999)]} = 0.1654.
\]

Note that in only 16.5% of the cases where the tests are positive will the person actually have the disease even though the test is 99% effective in detecting the disease when it is, in fact, present.
5. Let \((X_1, \ldots, X_n)\) be a random sample of an exponential random variable \(X\) with unknown parameter \(\lambda\). Determine the maximum-likelihood estimator of \(\lambda\).

\[
L(\lambda) = P(X_1, \ldots, X_n \mid \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda n \bar{X}}
\]

\[
\log L(\lambda) = n \log \lambda - \lambda n \bar{X}
\]

Equating \(d/d \lambda [\log L(\lambda)] = 0\), we get MLE \(\lambda = 1 / \bar{X}\)
6. Consider the random process \( Y(t) = (-1)^X(t) \), where \( X(t) \) is a Poisson process with rate \( \lambda \). Thus \( Y(t) \) starts at \( Y(0) = 1 \) and switches back and forth from +1 to -1 at random Poisson times \( T_i \).

(i) Find the mean of \( Y(t) \). \( Y(t) = 1 \) if \( X(t) \) is even; -1 if \( X(t) \) is odd.

\[
P(Y(t) = 1) = \exp(-\lambda t) \cos h \lambda t; \quad P(Y(t) = -1) = \exp(-\lambda t) \sin h \lambda t.
\]

Hence, 
\[
E[Y(t)] = \exp(-\lambda t) (\cos h \lambda t - \sin h \lambda t) = \exp(-2\lambda t).
\]

(ii) Find the autocorrelation function of \( Y(t) \).

\[
Y(t)Y(t+\tau) = 1 \text{ if there are an even number of events in (t, t+\tau); -1 otherwise.}
\]

\[
R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)] = \exp(-2\lambda \tau). \quad \text{Thus,} \quad R_Y(\tau) = \exp(-2\lambda |\tau|).
\]

(iii) Let \( Z(t) = A \cdot Y(t) \) where \( A \) is a discrete random variable independent of \( Y(t) \) and takes on values 1 and -1 with equal probability. Show that \( Z(t) \) is WSS.

\[
E[A] = 0; \quad E[A^2] = 1; \quad E[Z(t)] = E[A]E[Y(t)] = 0; \quad R_Z(\tau) = R_Y(\tau) = \exp(-2\lambda |\tau|).
\]

(iv) Find the power spectral density of \( Z(t) \).

\[
4 \lambda / (\omega^2 + 4 \lambda^2)
\]