

# NETWORK LAYER PERFORMANCE MODELING AND ANALYSIS

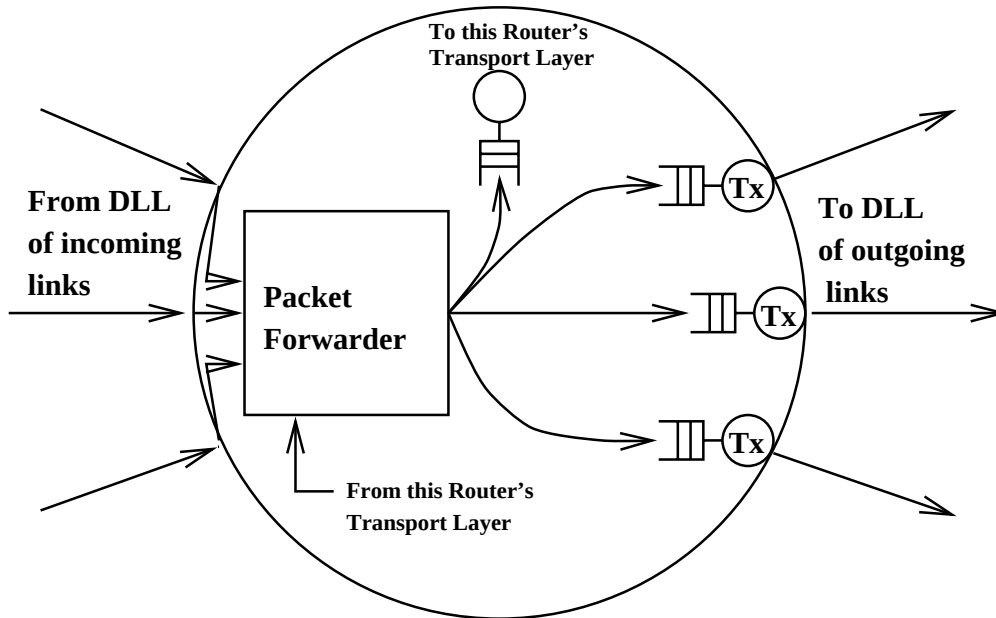
- Three parts.
- \* Part I: Essentials of Probability.
- Part II: Inside a Router.
- Part III: Network Analysis.

## Network Layer Performance Modeling and Analysis, Part I:

### ESSENTIALS OF PROBABILITY

- Motivation.
- Basic Definitions.
- Modeling Experiments with Uncertainty.
- Random Variables: Geometric, Poisson, Exponential.
- Read any of the probability references, e.g. Ross, Molloy, Papoulis, Stark and Woods.
- Check out WWW version of notes: <http://networks.ecse.rpi.edu/~vastola/pslinks/perf/node1.html>

## Motivation for Learning Probability in CCN



## Basic Definitions

- Think of probability as modeling an experiment.
- The set of all possible outcomes of the experiment is the sample space  $S$ .
- Classic “experiment”: Tossing a die.

$$S = \{1, 2, 3, 4, 5, 6\}$$

- Any subset  $A$  of  $S$  is an event, e.g.

$$A = \{the\ outcome\ is\ even\} = \{2, 4, 6\}$$

## Basic Operations on Events

- For any two events  $A, B$ , the following are also events
  - $\bar{A} = A$  complement = {all outcomes not in  $A$ }
  - $A \cup B = A$  union  $B =$  {all outcomes in  $A$  or  $B$  or both}
  - $A \cap B = A$  intersect  $B = AB =$  {all outcomes in  $A$  and  $B$ }
- Note  $\bar{\emptyset} = \emptyset$ , the empty set.
- If  $AB = \emptyset$ , then  $A$  and  $B$  are mutually exclusive.
- Can take many unions:  $A_1 \cup A_2 \cup \dots \cup A_n$
- Or even infinite unions:  $A_1 \cup A_2 \cup \dots = \bigcup_{n=1}^{\infty} A_n$
- Ditto for intersections.

## Probability on Events

- $P$  is a probability mass function if it maps each event  $A$  into a real number  $P(A)$  and
  - i)  $P(A) \geq 0$  for every event  $A \subseteq S$
  - ii)  $P(S) = 1$
  - iii) If  $A$  and  $B$  are mutually exclusive events, then  $P(A \cup B) = P(A) + P(B)$ . In fact, for any sequence of pair-wise-mutually-exclusive events,  $A_1, A_2, A_3, \dots$  (i.e.  $A_i A_j = \emptyset$  for any  $i \neq j$ ), we have
 
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

## Other Properties

- $P(\bar{A}) = 1 - P(A)$

- $P(A) \leq 1$

- $P(A \cup B) = P(A) + P(B) - P(AB)$

- $A \subseteq B \Rightarrow P(A) \leq P(B)$

## Conditional Probability

- $P(A | B) =$  (conditional) probability that the outcome is in  $A$  given that we know the outcome is in  $B$ .

$$P(A | B) = \frac{P(AB)}{P(B)} \quad P(B) \neq 0$$

- Example: Toss one die.

$$P(i = 3 | i \text{ is odd}) =$$

- Note that  $P(AB) = P(B)P(A | B) = P(A)P(B | A)$ .

## Independence

- Events  $A$  and  $B$  are independent if  $P(AB) = P(A)P(B)$ .
- Example: A card is selected at random from an ordinary deck of cards.  $A$  = event that the card is an ace.  $B$  = event that the card is a diamond.

$$P(AB) =$$

$$P(A) =$$

$$P(B) =$$

$$P(A)P(B) =$$

## Independence (cont.)

- Events  $A$  and  $B$  are independent if  $P(AB) = P(A)P(B)$ .
- Independence does NOT mean that  $A$  and  $B$  have “nothing to do with each other” or that  $A$  and  $B$  “have nothing in common”.
- Best intuition on independence is:

$A$  and  $B$  are independent if and only if  $P(A|B) = P(A)$  (equivalently,  $P(B|A) = P(B)$ ), i.e. if and only if knowing that  $B$  is true doesn't change the probability that  $A$  is true.

- Note: If  $A$  and  $B$  are independent and mutually exclusive, then  $P(A) = 0$  or  $P(B) = 0$ .

## Random Variables

- A random variable  $X$  maps each outcome  $s$  in the sample space  $S$  to a real number  $X(s)$ .

- Example: A fair coin is tossed 3 times.  $S = \{(TTT), (TTH), (THT), (HTT), (HTH), (THT), (HTT), (HTH), (HTH), (HHH)\}$ .

Let  $X$  be the number of heads tossed in 3 tries.

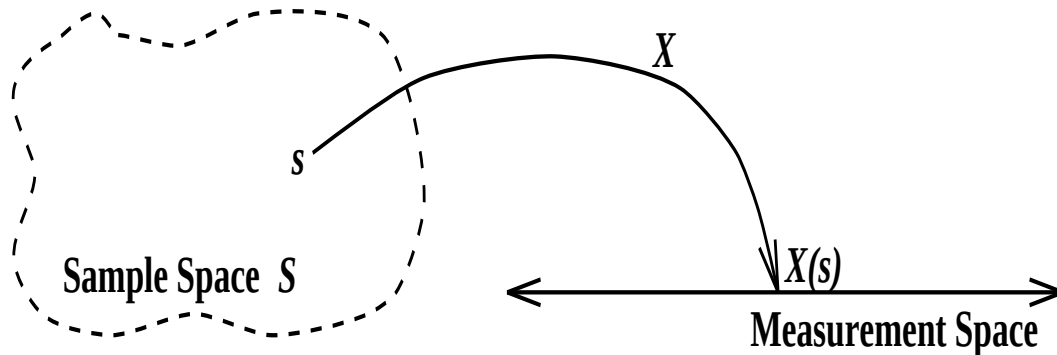
- So  $P(X=0) = X(TTT) =$   
 $X(TTH) =$   
 $X(THT) =$   
 $X(HTT) =$   
 $X(HTH) =$   
 $X(THT) =$   
 $X(HTH) =$   
 $X(HHT) =$   
 $P(X=1) =$   
 $P(X=2) =$   
 $P(X=3) =$

## Random Variable as a Measurement

- Think of much more complicated experiments
  - A chemical reaction.
  - A laser emitting photons.
  - A packet arriving to an router.
- We cannot give an exact description of a sample space in these cases, but we can still describe specific measurements on them
  - The temperature change produced.
  - The number of photons emitted in one millisecond.
  - The time of arrival of the packet.

## Random Variable as a Measurement (cont.)

- Thus a random variable can be thought of as a measurement on an experiment.



## Probability Mass Function for a Random Variable

- The probability mass function (PMF) for a (discrete-valued) random variable  $X$  is

$$P_X(x) = P(X = x) = P(\{s \in S \mid X(s) = x\})$$

- Note that  $P_X(x) \geq 0$  for  $-\infty < x < \infty$ .
- Also for a (discrete-valued) random variable  $X$

$$\sum_{x=-\infty}^{\infty} P_X(x) = 1$$

## Cumulative Distribution Function

- The cumulative distribution function (CDF) for a random variable  $X$  is

$$F_X(x) = P(X \leq x) = P(\{s \in S \mid X(s) \leq x\})$$

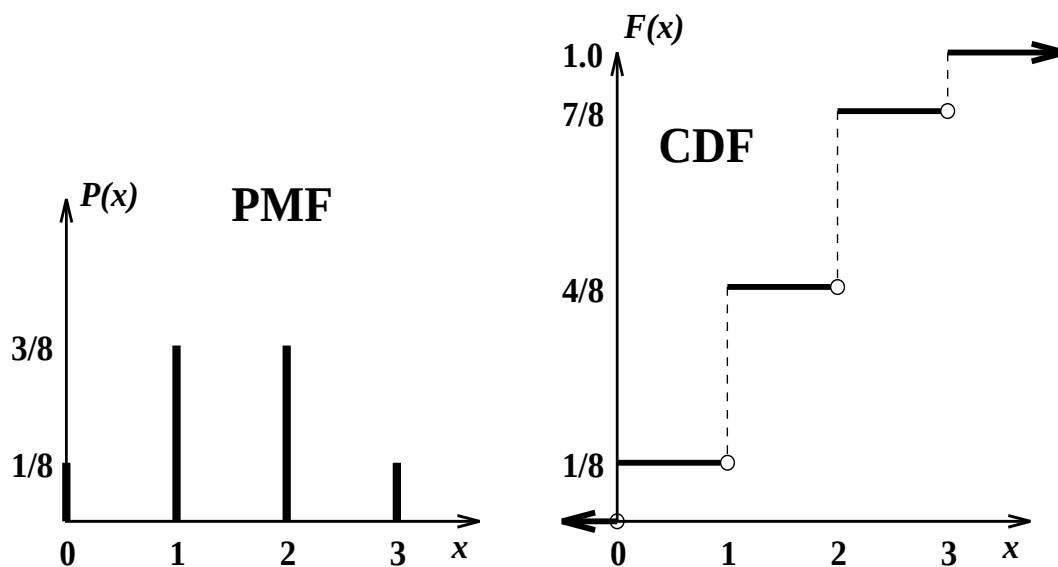
- Note that  $F_X(x)$  is non-decreasing in  $x$ , i.e.

$$x_1 \leq x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$$

- Also

$$\lim_{x \rightarrow -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} F_X(x) = 1$$

## PMF and CDF for the 3 Coin Toss Example



## Expectation of a Random Variable

- The expectation (average) of a (discrete-valued) random variable  $X$  is

$$\bar{X} = E(X) = \sum_{x=-\infty}^{\infty} x P(X = x) = \sum_{x=-\infty}^{\infty} x P_X(x)$$

- Three coins example:

$$E(X) = \sum_{x=0}^3 x P_X(x) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$

$$= 1.5$$

## Important Random Variables: Bernoulli

- The simplest possible measurement on an experiment: Success ( $X = 1$ ) or failure ( $X = 0$ ).
- Usual notation:  
 $P^X(1) = P(X = 1) = p$        $P^X(0) = P(X = 0) = 1 - p$
- $E(X) =$

### Important Random Variables: Binomial

- Let  $X$  = the number of success in  $n$  independent Bernoulli experiments (or trials).

$$P(X = 0) =$$

$$P(X = 1) =$$

$$P(X = 2) =$$

- In general,  $P(X = x) =$

- Exercise: Show that

$$\sum_n^{x=0} P^X(x) = 1 \quad \text{and} \quad E(X) = np$$

### Important Random Variables: Geometric

- Let  $X$  = the number of independent Bernoulli trials until the first success.

$$P(X = 1) = p$$

$$P(X = 2) = (1 - p)p$$

$$P(X = 3) = (1 - p)^2 p$$

- In general,  $P(X = x) = (1 - p)^{x-1} p$  for  $x = 1, 2, 3, \dots$

- Exercise: Show that

$$\sum_{x=1}^{\infty} P^X(x) = 1 \quad \text{and} \quad E(X) = \frac{1}{p}$$

## Important Random Variables: Poisson

- A Poisson random variable  $X$  is defined by its PMF

$$P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \quad x = 0, 1, 2, \dots$$

where  $\lambda > 0$  is a constant.

- Exercise: Show that

$$\sum_{x=0}^{\infty} P^X(x) = 1 \quad \text{and} \quad E(X) = \lambda$$

- Poisson random variables are good for counting things like the number of customers that arrive to a bank in one hour or the number of packets that arrive to an router in one second.

## Continuous-valued Random Variables

- So far we have focused on discrete(-valued) random variables, e.g.  $X(s)$  must be an integer.
- Examples of discrete random variables: number of arrivals in one second, number of attempts until success.
- A continuous-valued random variable takes on a range of real values, e.g.  $X(s)$  ranges from 0 to  $\infty$  as  $s$  varies.
- Examples of continuous(-valued) random variables: time when a particular arrival occurs, time between consecutive arrivals.

## Continuous-valued Random Variables (cont.)

- A discrete random variable has a “staircase” CDF.
- A continuous random variable has (some) continuous slopes to its CDF.

- Thus, for a continuous random variable  $X$ , we can define its probability density function (pdf)

$$f_X(x) = \frac{dF_X(x)}{dx}$$

- Note that since  $F_X(x)$  is non-decreasing in  $x$ , we have

$$f_X(x) \geq 0 \quad \text{for all } x.$$

## Properties of Continuous Random Variables

- From the Fundamental Theorem of Calculus, we have

$$F_X(x) = \int_x^{-\infty} f_X(x) dx$$

- In particular,

$$\int_{-\infty}^{\infty} f_X(x) dx = F_X(\infty) = 1$$

- More generally,

$$\int_b^a f_X(x) dx = F_X(a) - F_X(b) = P(a < X \leq b)$$

## Expectation of a Continuous Random Variable

- The expectation (average) of a continuous random variable  $X$  is given by

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Note that this is just the continuous equivalent of the discrete expectation

$$E(X) = \sum_{x=-\infty}^{\infty} x P_X(x)$$

## Important Continuous Random Variable: Exponential

- Used to represent time, e.g. until next arrival.
- Has pdf
- Show

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

for some  $\lambda > 0$

$$\int_{-\infty}^{\infty} f_X(x) dx = 1 \quad \text{and} \quad E(X) = \frac{1}{\lambda}$$

**Need to use Integration by Parts!**

## Exponential Random Variable (cont.)

- The CDF of an exponential random variable is

$$F_X(x) = \int_x^0 f_X(\hat{x}) d\hat{x} = \int_x^0 \lambda e^{-\lambda \hat{x}} d\hat{x}$$

$$= \left[ -e^{-\lambda \hat{x}} \right]_x^0 = 1 - e^{-\lambda x}$$

- So  $P(X > x) = 1 - F_X(x) = e^{-\lambda x}$

## Memoryless Property of the Exponential

- An exponential random variable  $X$  has the property that “the future is independent of the past” i.e. the fact that it hasn’t happened yet, tells us nothing about how much longer it will take.
- In math terms

$$P(X > s + t \mid X > t) = P(X > s) \quad \text{for } s, t > 0$$

## Memoryless Property of the Exponential (cont.)

$$\text{Proof: } P(X > s + t \mid X > t) = \frac{P(X > s + t, X > t)}{P(X > t)}$$

$$= \frac{P(X > t)}{P(X > t)}$$

$$= \frac{e^{-\lambda t}}{e^{-\lambda(s+t)}}$$

$$= e^{-\lambda s}$$

$$= P(X > s)$$