# ECSE 6460: Multivariable Control Systems

Homework set 2. Due date: 2 October 2009

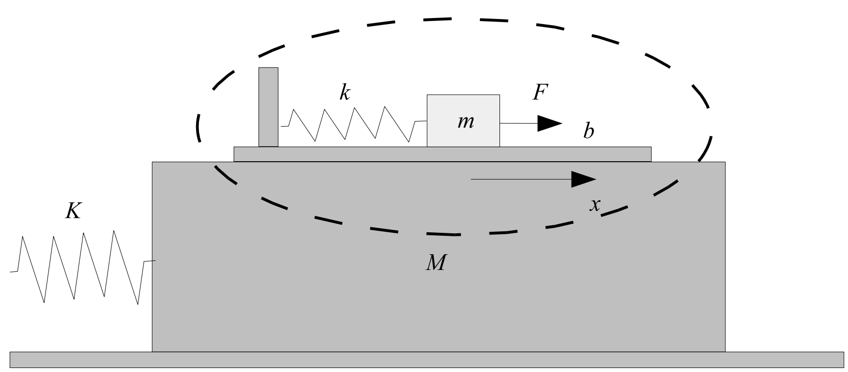
**Points:** Problem 1 = 15+20+20 pts, Problem 2 = 10+20 pts, Problem 3 = 15 pts

**Problem 1.** This is a three-part problem. Consider the standard mass-spring-damper position control problem shown in the circle below. We want to control $x$, which is the position of the (small) mass with respect to its at-rest position. The symbols $m$, $k$, and $b$ represent the mass, spring constant, and friction coefficient in this problem. For control input, we can apply a force $F$ to the mass. The non-standard part of this problem comes from the fact that this control system is fixed on top of a flexible structure (e.g. a swing). Upon linearization, the flexible structure can be modelled as a (much larger) mass $M$ and another spring with spring constant $K$.

The parameters of the problem are given as follows:

$$\begin{aligned} m &= 0.1\,\mathrm{kg},\ k = 1\,\mathrm{N\,m^{-2}},\ b = 1\,\mathrm{N\,m^{-1}\,s}, \\ M &= 1000\,\mathrm{kg}, K = 10^5\,\mathrm{N\,m^{-2}}. \end{aligned}$$

It is also known that the amplitude of the reference signal does not exceed 10 cm. Further, the worst case disturbance can be modelled as an impulse force acting on the mass $M$ such that the flexible structure vibrates (the base) with an amplitude of 40 cm. Notice that $x$ is measured in the coordinate frame that is moving with the big mass.




Task: Design control systems that can meet the following performance criteria:

- For reference tracking: the rise time is less than 0.3 second, and the overshoot is less than 5%,

- Disturbance rejection: the worst case disturbance does not contribute to more than 2cm of displacement (amplitude < 2cm).

(a) Use 2 degree of freedom technique with disturbance feedforward controller. Assume that you can measure the inertia force acting on the small mass due to the disturbance (for example from an accelerometer placed on the big mass) and use this information for controller design.
(b) Use 2 degree of freedom technique with the prefilter $K_r$ (see the textbook and example in class).
(c) Use $H_\infty$ controller synthesis.
For each design, include a MATLAB analysis/simulation. Motivate your design (not only the final solution). Hint: When analyzing the dynamics of the flexible structure, you can ignore the mass of the control system. For numerical stability, if necessary, you can model a very small friction for the base.

**Problem 2.** Consider the real valued matrix $G \in \mathbb{R}^{n \times n}$, which can be thought of as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$

$$y = Gd, \; y, d \in \mathbb{R}^n.$$

We are interested in finding the maximum gain of $G$, which is given by:

$$\sigma_{\max} = \max_{d \in \mathbb{R}^n} \frac{\|Gd\|}{\|d\|},$$

where $\|\cdot\|$ is the Euclidian norm.
(a) Show that

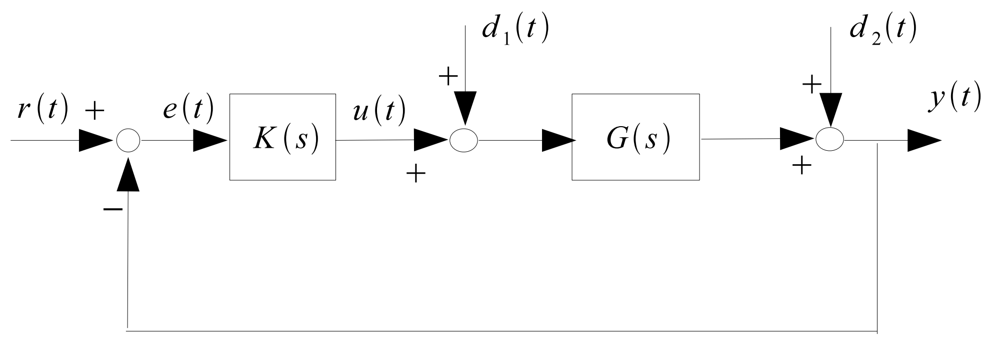
$$\sigma_{\max} = \max_{\|d\|=1} \|Gd\|.$$

(b) Prove that $\sigma_{\max}$ is the largest eigenvalue of $G^T G$. Hint: See (a) as a quadratic optimization problem with an equality constraint.

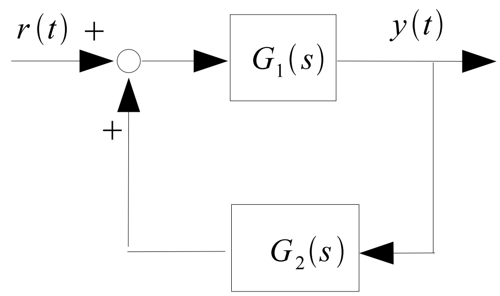
**Problem 3.** Consider the 1-DOF feedback control loop below. Compute the transfer matrix from all inputs $(r, d_1, d_2)$ to all outputs $(e, u, y)$. In particular, demonstrate your calculation by using the positive feedback rule. See the positive feedback loop below. The positive feedback rule says

$$Y(s) = (I - G_1 G_2)^{-1} G_1 \cdot R(s).$$

Hint: I want you to identify $G_1$ and $G_2$ in those cases.



$r(t)$
+
$e(t)$
$K(s)$
$u(t)$
+
$d_1(t)$
+
$G(s)$
$d_2(t)$
+
+
$y(t)$
−

$r(t)$
+
$G_1(s)$
$y(t)$
+
$G_2(s)$