Model Reduction

Consider state-space representation:

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\right\} \text{minimal representation}$$

$$G(s) = C(sI - A)^{-1}B + D$$

We partition the state space into $x = \begin{bmatrix} x_1 \\ \hdashline x_2 \end{bmatrix}$, thus:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u$$

$$\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2 u$$

$$y = C_1 x_1 + C_2 x_2 + Du$$

Assume that all poles are in the CLHP

There are two ways to reduce the model by ignoring x_2

Truncation: Basically, here we just ignore x_2, so:

$$\dot{x}_1 = A_{11}x + B_1 u$$

$$y = C_1 x_1 + Du$$

Thus:

$$\boxed{G_a(s) = C_1(sI - A_1)^{-1}B_1 + D}$$

Not much can be said about $(G(s) - G_a(s))$ except that

$$G_a(\infty) = G(\infty) = D$$

This is because x_1 and x_2 are coupled. If they are decoupled, then $A_{12} = 0$, $A_{21} = 0$, thus:

$$(sI - A)^{-1} = \begin{bmatrix} sI - A_{11} & 0 \\ 0 & sI - A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} (sI - A_{11})^{-1} & 0 \\ 0 & (sI - A_{22})^{-1} \end{bmatrix}$$

$$G(s) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} (sI - A_{11})^{-1} & 0 \\ 0 & (sI - A_{22})^{-1} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D$$

$$= C_1(sI - A_{11})^{-1}B_1 + C_2(sI - A_{22})^{-1}B_2 + D$$

Thus: $G(s) - G_a(s) = C_2 (sI - A_{22})^{-1} B_2$

A special case: A is diagonal: $A = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

Suppose that we only keep the first m states. Then:

$$A_{22} = \begin{bmatrix} \lambda_{m+1} & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \rightarrow (sI - A_{22})^{-1} = \begin{bmatrix} \frac{1}{s-\lambda_{m+1}} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix}$$

$$G(s) - G_a(s) = \begin{bmatrix} C_{21} & C_{22} & \cdots & C_{2J} \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_{m+1}} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} B_{21} \\ \vdots \\ B_{2J} \end{bmatrix}$$

$\boxed{J \triangleq (n-m)}$

$$= \sum_{K=1}^{J} \frac{C_{2K}^T B_{2K}}{s - \lambda_{m+K}}$$

$$\|G(s) - G_a(s)\|_\infty \leq \sum_{K=1}^{J} \frac{\sigma_{max}(C_{2K}^T B_{2K})}{|\mathrm{Re}\, \lambda_{m+K}|}$$

Quick guideline: Ignoring $C^T B$ part, it is better to discard poles that are faster ($|\mathrm{Re}\, \lambda|$ large)

Residualization

In this technique, we assume that $\dot{x}_2 = 0$, i.e:

$\dot{x}_2 = A_{21} x_1 + A_{22} x_2 + B_2 u = 0$

$x_2 = -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u$, substitute into $\dot{x}_1$:

$$\dot{x}_1 = A_{11} x_1 - A_{12} A_{22}^{-1} A_{21} x_1 + (B_1 - A_{12} A_{22}^{-1} A_{21}) u$$
$$Y = C_1 x_1 + D$$

We can show that in this case : $G(0) = G_a(0) = -CA^{-1}B + D$

This approximation is suitable if the dynamics of x_2 is much faster than that of x_1 (see singular perturbation)

Notice that for decoupled x_1 and x_2, residualization and truncation are the same

Balanced Realization

Define: $P \triangleq \int_0^\infty e^{At} BB^T e^{A^T t} \, dt$ is the controllability Grammian

$Q \triangleq \int_0^\infty e^{A^T t} C^T C e^{At} \, dt$ is the observability Grammian

They can be computed through :

$$AP + PA^T + BB^T = 0$$
$$A^T Q + QA + C^T C = 0$$

In balanced realization, the state space is linearly transformed such that $P = Q$ is a diagonal matrix.

Definition : The Hankel singular values of a stable system $G(s)$ are the positive roots of $\lambda(PQ)$

Balanced Truncation

Truncate the states corresponding to the smallest Hankel singular values

Suppose that we keep only the first m largest Hankel singular values, then

$$\|G(s) - G_a(s)\|_\infty \leq 2(\sigma_{m+1} + \sigma_{m+2} + \cdots + \sigma_n)$$

where σ_i are the Hankel singular values.