Structured Singular Value μ

We know that it is generally harder to destabilize the $M\Delta$ loop with structured perturbation than unstructured perturbation

Definition: The structured singular value $\mu(M)$ is defined as:

$$\frac{1}{\mu(M)} = \min_{\Delta} \{ \sigma_{max}(\Delta) \mid \det(I - M\Delta) = 0 \}, \text{ structured } \Delta$$

Generally speaking, $\boxed{M \text{ is a complex valued matrix}}$

Thus, the smaller $\mu(M)$ is, the better for robustness. Let us consider two extreme cases.

Least Restrictive Structure

If Δ is complex and without any structure,

$$\mu(M) = \sigma_{max}(M)$$

Proof: Perform SVD on M : $M = U \Sigma V^H$, define

$$\Delta = V \begin{pmatrix} \frac{1}{\sigma_{max}(M)} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} U^H, \text{ thus } \sigma_{max}(\Delta) = \frac{1}{\sigma_{max}(M)}$$

$$M\Delta = U \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} U^H = U \begin{pmatrix} 1 & & \\ & 0 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} U^{-1}, \text{ which means that } 1 \text{ is}$$

an eigenvalue of $M\Delta$. Therefore $\mu(M) \geq \sigma_{max}(M)$

To proof the opposite, notice that if 1 is an eigenvalue of $M\Delta$, then $\exists x \neq 0$ $M\Delta x = x$. But then we have:

$$\|x\| = \|M\Delta x\| \leq \sigma_{max}(M)\, \sigma_{max}(\Delta)\, \|x\|, \text{ or}$$

$\sigma_{max}(M) \cdot \sigma_{max}(\Delta) \geq 1$

$\frac{1}{\sigma_{max}(\Delta)} \leq \sigma_{max}(M)$, which implies $\mu(M) \leq \sigma_{max}(M)$ (q.e.d)

Most Restrictive Case

If Δ is a complex scalar, then $\mu(M) = \rho(M)$

Proof: Perform eigenvalue decomposition on M:

$$M = U \Lambda U^{-1}$$

Define $\Delta = \frac{1}{\lambda_{max}}$, thus $M\Delta$ has an eigenvalue at 1. Therefore

$$\mu(M) \geq |\lambda_{max}| = \rho(M)$$

To prove the opposite, if 1 is an eigenvalue of $M\Delta$, then $\exists x \neq 0$ such that $M\Delta x = x$, but since Δ is scalar, this means that x is an eigenvector of M with eigenvalue $\frac{1}{\Delta}$, the smallest Δ that has this property is $\frac{1}{\lambda_{max}}$

Thus; for any structured singular value:

$$\boxed{\rho(M) \leq \mu(M) \leq \sigma_{max}(M)}$$

Example: $M = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$

With unstructured uncertainty, we can find the smallest Δ that makes $\det(I - M\Delta) = 0$ as $\Delta = \begin{bmatrix} 0.2 & -0.1 \\ 0.2 & -0.1 \end{bmatrix}$ with $\sigma_{max}(\Delta) = \frac{1}{\sigma_{max}(M)} = \frac{1}{3.162}$

For scalar Δ, we can find the smallest Δ that makes $\det(I - M\Delta) = 0$ as $\Delta = \frac{1}{\lambda_{max}(M)} = 1$, since $I - M = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \rightarrow \det = 0$

So we expect that for any other structured uncertainty

$$1 \leq \mu(M) \leq 3.162$$

Consider the diagonal uncertainty $\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$, we can compute

that the smallest Δ that makes $\det(I - M\Delta) = 0$ is

$$\Delta = \begin{bmatrix} 1/3 & 0 \\ 0 & -1/3 \end{bmatrix},$$

thus $\sigma_{max}(\Delta) = \frac{1}{3}$, $\mu(M) = 3$

In MATLAB, use the function 'mu' to approximate $\mu(M)$

Often, $\min_D \sigma_{max}(DMD^{-1})$ is used as upperbound for $\mu(M)$

If there are three or fewer blocks, this upperbound is tight

Relation to Robust Stability with structured uncertainty

Recall that RS means:

$\forall \omega, \forall \Delta$, $\det(I - M\Delta) \neq 0$

Therefore for structured uncertainty:

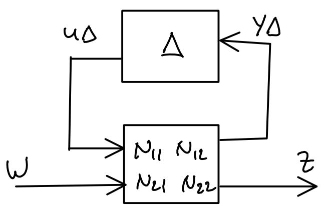
Thm: Assume that the plant is NS and Δ is stable, then the $M\Delta$ block is stable for all allowed uncertainty block Δ, $\sigma_{max}(\Delta) \leq 1$ iff

$$\forall \omega, \ \mu(M(j\omega)) < 1$$

In MATLAB, use 'ssv' to produce structured singular value plot.

Robust Performance

We can use μ to assess the robustness property of the performance



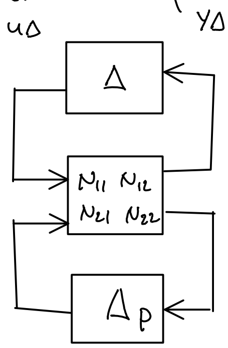
Recall that RP means:

NS and $\|F\|_\infty < 1$ for all $\|\Delta\|_\infty \leq 1$

where F is the TF from w to z

$$F = F_u(N, \Delta)$$

Theorem: RP if and only if

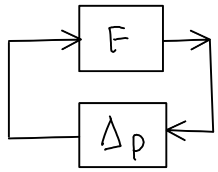


is robustly stable for any allowed Δ, $\|\Delta\|_\infty \leq 1$ and any Δ_p (unstructured), $\|\Delta_p\|_\infty \leq 1$

$$\hat{\Delta} = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_p \end{bmatrix}, \quad \|\hat{\Delta}\|_\infty \leq 1$$

note that $\hat{\Delta}$ is structured

Proof: Write RP condition as RS for



with unstructured Δ_p, $\|\Delta_p\|_\infty \le 1$.

Then rewrite F with the $N\Delta$ loop, and then regroup the uncertainty.

Observe that, just as in the SISO case, NS and RP implies RS.