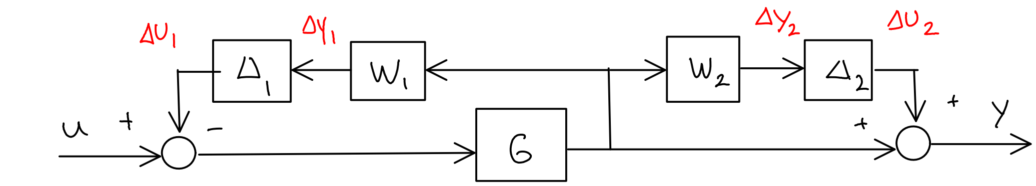
MIMO uncertainty and Robustness

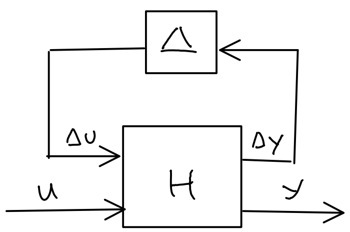
A couple of factors that differentiate the MIMO case from SISO case:
- structure of uncertainty
- input and output uncertainty

Structure of uncertainty

Consider the example:



We can lump the uncertainties as:



where $\Delta(s) = \begin{bmatrix} \Delta_1(s) & 0 \\ 0 & \Delta_2(s) \end{bmatrix}$

Observe that $\|\Delta_1\|_\infty \leq 1, \|\Delta_2\|_\infty \leq 1 \Leftrightarrow \|\Delta\|_\infty \leq 1$

$\Delta(s)$ is a 2x2 system. However it is structured as a diagonal system. If we ignore this constrained structure, we could get conservative result in robustness analysis.

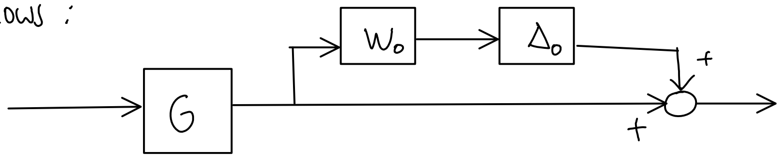
The term unstructured uncertainty refers to the uncertainty block $\Delta(s)$ with only $\|\Delta\|_\infty \leq 1$ constraint.

Input and Output uncertainty

Because transfer matrix multiplications generally do not commute, we have e.g.

$$G\underbrace{(I+W\Delta)}_{\text{input}} \neq \underbrace{(I+W\Delta)}_{\text{output}}G$$

Output side uncertainty can be modeled as unstructured uncertainty as follows:



$$G_p(s) = \left(I + W_O(s)\,\Delta_o(s)\right) G(s)$$, where $W_O(s)$ is scalar and $\|\Delta_o\|_\infty \le 1$

The weight $W_O(s)$ can be constructed as follows:

$$\forall \omega\ \forall G_p \in \Pi, |W_o(j\omega)| \ge \sigma_{max}\left[\left(G_p(j\omega) - G(j\omega)\right) G^{-1}(j\omega)\right]$$

Similarly, on the input side:

$$G_p(s) = G(s)\left(I + W_I(s)\,\Delta_I(s)\right)$$, where $W_I(s)$ is a scalar and $\|\Delta_I\|_\infty \le 1$

The weight $W_I(s)$ can be constructed as follows:

$$\forall \omega\ \forall G_p \in \Pi,\ |W_I(j\omega)| \ge \sigma_{max}\left[G^{-1}(j\omega)\left(G_p(j\omega) - G(j\omega)\right)\right]$$

If $G(j\omega)$ is not invertible, then we can use pseudo inverse. Note that there are geometric considerations.

Ex: Given a 1x2 plant with uncertainty:

$$G_p(s) = \left[\, G_1(s)\left(1 + W_1(s)\Delta_1(s)\right) \quad G_2(s)\left(1 + W_2(s)\Delta_2(s)\right) \right]$$

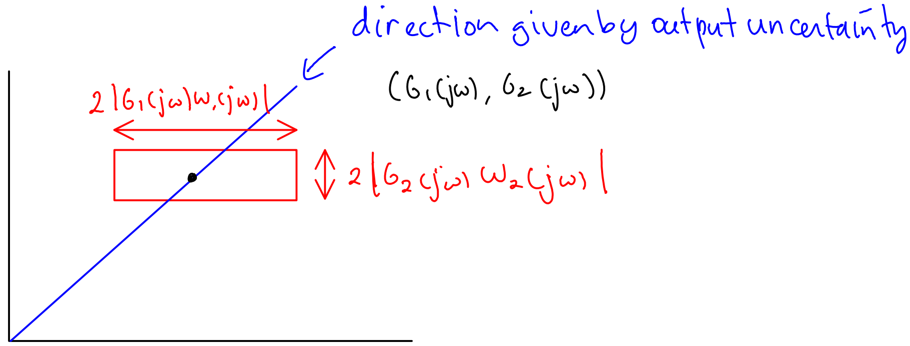
To model it as output side uncertainty, we have:

$$G_p(s) = \left(1 + W_O(s)\,\Delta_O(s)\right) \underbrace{\left[\, G_1(s) \quad G_2(s) \right]}_{G(j\omega)}$$

Thus, at every ω,

$$\left(G_p(j\omega) - G(j\omega)\right) = W_o(j\omega)\,\underbrace{\Delta_o(j\omega)}_{1\times 1 \text{ transfer function}}\,G(j\omega)$$

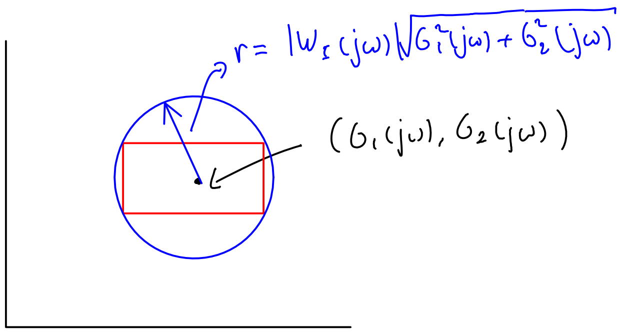
Illustration:



On the other hand, if we model it as unstructured input uncertainty:

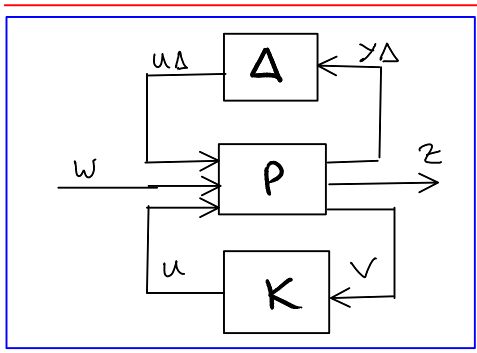
$$G_p(s) = [G_1(s) \quad G_2(s)] \cdot (I + W_I(s)\,\Delta_I(s))$$

$$\forall\omega,\quad (G_p(j\omega) - G(j\omega)) = W_I(j\omega)\cdot\begin{bmatrix} G_1(j\omega) & G_2(j\omega)\end{bmatrix}\begin{bmatrix}\Delta_{11}(j\omega) & \Delta_{12}(j\omega) \\ \Delta_{21}(j\omega) & \Delta_{22}(j\omega)\end{bmatrix}$$

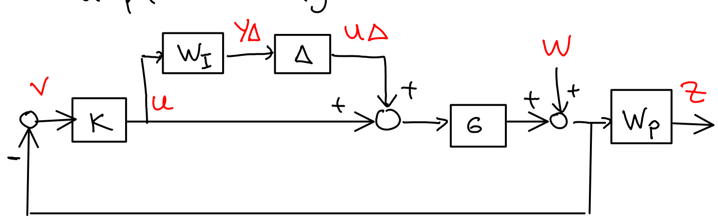


Thus, input side uncertainty can capture the uncertainty with some conservativeness

General Control Problem Formulation:

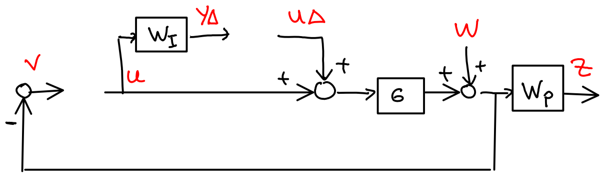


Example: See Fig 8.7 in text book



General Plant model:

$$\begin{bmatrix} y_\Delta \\ z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33}\end{bmatrix}\begin{bmatrix} u_\Delta \\ w \\ u \end{bmatrix}$$



$$\left.\begin{aligned} y_\Delta &= W_I\, u \\ z &= W_p(w + Gu + G u_\Delta) \\ v &= -(w + Gu + G u_\Delta) \end{aligned}\right\} P = \begin{bmatrix} 0 & 0 & W_I \\ W_pG & W_p & W_pG \\ -G & -I & -G \end{bmatrix}$$

Once we have a controller, the lower loop can be closed to assess robustness properties. Use the lower LFT

$$P_{11} \triangleq \begin{bmatrix} 0 & 0 \\ W_pG & W_p \end{bmatrix}; \; P_{12} \triangleq \begin{bmatrix} W_I \\ W_pG \end{bmatrix}; \; P_{21} = \begin{bmatrix} -G & -I \end{bmatrix}; \; P_{22} = -G$$

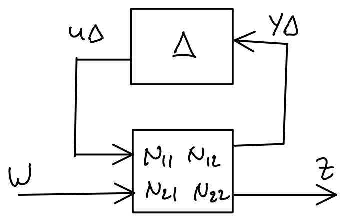
$$F_\ell(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \triangleq N$$

$$N = \begin{bmatrix} 0 & 0 \\ W_pG & W_p \end{bmatrix} + \begin{bmatrix} W_I \\ W_pG \end{bmatrix} K(I+GK)^{-1}\begin{bmatrix} -G & -I \end{bmatrix}$$

$$N = \begin{bmatrix} -W_IK(I+GK)^{-1}G & -W_IK(I+GK)^{-1} \\ W_pG - W_pGK(I+GK)^{-1}G & W_p - W_pGK(I+GK)^{-1} \end{bmatrix}$$

$$N = \begin{bmatrix} -W_IKG(I+KG)^{-1} & -W_IK(I+GK)^{-1} \\ W_pG(I - KG(I+KG)^{-1}) & W_p(I - GK(I+GK)^{-1}) \end{bmatrix}$$

$$N = \begin{bmatrix} -W_IKG(I+KG)^{-1} & -W_IK(I+GK)^{-1} \\ W_pG(I+KG)^{-1} & W_p(I+GK)^{-1} \end{bmatrix}$$



Recall that good performance is characterized by the transfer function from w to z being small.

In this setup:

Nominal Stability : N is internally stable

Nominal Performance : $\|N_{22}\|_\infty < 1$, and NS

Robust Stability & Robust Performance

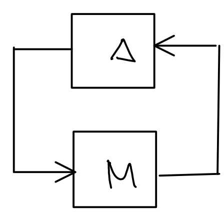
For any Δ, if we close the upper loop (using upper LFT), we get the TF from w to z:

$$F \triangleq F_u(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1} N_{12}$$

If the system is nominally stable, to check robust stability we only need to make sure that

$$\boxed{(I - N_{11}\Delta)^{-1} \text{ is stable}}$$

In the book, N_{11} is called M, thus:



needs to be stable

for any $\|\Delta\|_\infty \leq 1$

For robust performance : $\|F\|_\infty < 1$ for all $\|\Delta\|_\infty \leq 1$, and NS