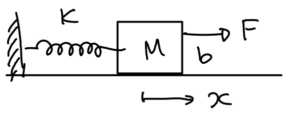
# SISO uncertainty and Robustness

Main idea: because of modeling limitation, there is always some uncertainty in the plant model. Thus, instead of a single unique plant model $G(s)$, we have a family of plants $\Pi$, i.e

any $G_p \in \Pi$ is a possible plant model

Example: mass-spring-damper system




Suppose that $b$ is not known precisely, but given in a range $b_{min} \leq b \leq b_{max}$

Example: high frequency modes of mechanical systems are typically ignored and can be a source of uncertainty.

Challenges:

- How to model uncertainty
- How to make sure that close loop stability is retained under uncertainty
- How to make sure that desired performance is retained

In modeling the uncertainty, we want to separate the uncertainty from the plant model. Identify a plant model $G \in \Pi$ as the <u>nominal model</u>, and other element $G_p \in \Pi$ is a perturbation of $G$.

Example: Suppose that $G_p(s) = K\, G_o(s)$, where $K_{min} \leq K \leq K_{max}$

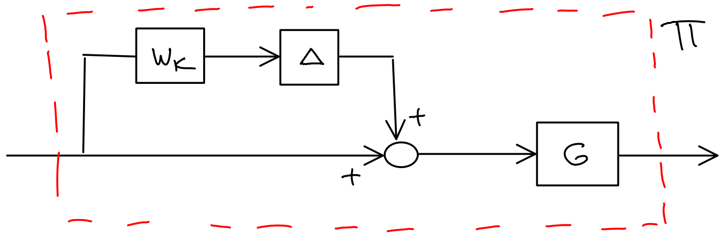
Define: $\bar{K} = \dfrac{K_{min} + K_{max}}{2}$ ; $W_K = \dfrac{K_{max} - K_{min}}{2\bar{K}}$

Thus: any $K \in [K_{min}, K_{max}]$ can be written as $\bar{K}(1 + W_K \Delta)$ for some $-1 \leq \Delta \leq 1$.

Therefore any $G_p(s) = K\, G_o(s) = \bar{K}\, G_o(s)(1 + W_K \Delta)$

We can define the nominal model $G = \bar{k} G_0$, and $G_p = G(1+W_k \Delta)$

Thus the whole family of uncertain models can be represented as




$\Pi$ for any $|\Delta| \le 1$

multiplicative uncertainty

Another example: $G_p(s) = (1 + s\tau) G_o(s)$ with $\tau_{min} \le \tau \le \tau_{max}$

can be modeled as multiplicative uncertainty

$\bar{\tau} = \dfrac{\tau_{min} + \tau_{max}}{2}$ ; $W_\tau = \dfrac{\tau_{max} - \tau_{min}}{2\bar{\tau}}$, thus $\tau = \bar{\tau}(1 + W_\tau \Delta)$, $|\Delta| \le 1$

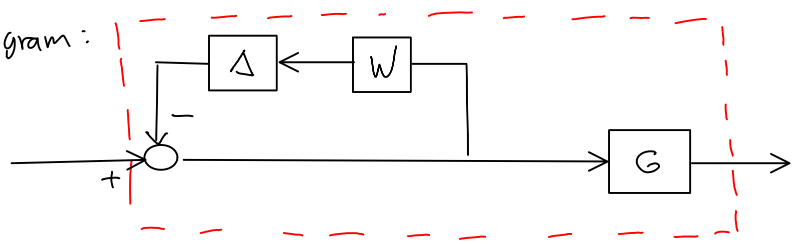
$$G_p(s) = (1 + s\bar{\tau}(1 + W_\tau \Delta))\, G_o(s) = G_o + G_o s \bar{\tau} + G_o s \bar{\tau} W_\tau \Delta$$
$$= G(1 + W_I \Delta)$$

$$G = G_o + G_o s \bar{\tau}, \quad W_I = \frac{G_o + G_o s\bar{\tau} + G_o s \bar{\tau} W_\tau \Delta - G}{G \Delta}$$

$$= \frac{\cancel{G_o} s \bar{\tau} W_\tau}{\cancel{G_o}(1 + s\bar{\tau})} = \frac{s\bar{\tau} W_\tau}{1 + s\bar{\tau}}$$

$\boxed{\text{Inverse Multiplicative uncertainty:}}$ $G_p = \dfrac{G}{1 + W\Delta}$

Block diagram:




Example: $G_p(s) = \dfrac{1}{\tau s + 1} G_o(s)$ ; $\tau_{min} \le \tau \le \tau_{max}$

can be modeled as inverse multiplicative uncertainty:

$\bar{\tau} = \dfrac{\tau_{min} + \tau_{max}}{2}$ ; $W_\tau = \dfrac{\tau_{max} - \tau_{min}}{2\bar{\tau}}$, thus $\tau = \bar{\tau}(1 + W_\tau \Delta)$

$$G_p(s) = \frac{G_0(s)}{\bar{\tau}s+1}\,\frac{\bar{\tau}s+1}{s\bar{\tau}(1+W_\tau\Delta)+1} = \frac{G}{1+W\Delta}, \text{ thus:}$$

$$G = \frac{G_0(s)}{\bar{\tau}s+1}, \quad 1+W\Delta = \frac{s\bar{\tau}+s\bar{\tau}W_\tau\Delta+1}{s\bar{\tau}+1}, \quad W = \frac{s\bar{\tau}W_\tau}{s\bar{\tau}+1}$$

Another example: Pole uncertainty: $G_p(s) = \frac{1}{s+p} G_0(s)$, $p_{min} \le p \le p_{max}$

Similarly as above: $\bar{p} = \frac{p_{min}+p_{max}}{2}$, $W_{ip} = \frac{p_{max}-p_{min}}{2\bar{p}}$

$$G_p(s) = \frac{G_0(s)}{s+\bar{p}} \cdot \frac{s+\bar{p}}{s+\bar{p}(1+W_{ip}\Delta)} \Rightarrow G = \frac{G_0(s)}{s+\bar{p}}$$

$$1+W\Delta = \frac{s+\bar{p}+\bar{p}W_{ip}\Delta}{s+\bar{p}}, \text{ thus } W = \frac{\bar{p}W_{ip}}{s+\bar{p}}$$

As opposed to parametric uncertainty, we can also model unstructured uncertainty in the frequency domain:

Idea:

1. Select a nominal model $G(s)$
2. sample the frequency range for a set of frequency points, $\omega_i$, $i=1\ldots N_s$
3. for each $\omega_i$, compute or find an upper bound for

$$r_i = \sup\{\,|G_p(j\omega_i) - G(j\omega_i)| \;|\; G_p \in \Pi\}$$

Example: $G_p(s) = \frac{K}{\tau s+1} e^{-\theta s}$, $2 \le K, \theta, \tau \le 3$

See Figure 7.2 in the book

Two ways to represent the uncertainty:

- Additive uncertainty: $G_p(s) = G(s) + W_A(s)\Delta_A(s)$ where $|\Delta_A(j\omega)| \leq 1$
  $W_A(s)$ can be chosen such that $\forall i \in \{1, 2, \dots N_s\}$
  $$|W_A(j\omega_i)| \geq r_i$$

- Multiplicative uncertainty: $G_p(s) = G(s)\,(1 + W_M(s)\Delta_M(s))$
  where $|\Delta_M(j\omega)| \leq 1$, $\forall \omega$
  $W_M(s)$ can be chosen such that $\forall i \in \{1, 2, \dots N_s\}$
  $$|W_M(j\omega_i)| \geq \frac{r_i}{|G(j\omega_i)|}$$

Multiplicative uncertainty is sometime more informative, since $|W_M(j\omega)| \geq 1$ implies that the origin is included in the uncertainty disc, implying that $G(s)$ might have a zero at that particular frequency. Therefore good tracking at that frequency is not possible.

See Example 7.6:

<u>Unmodeled Dynamics: Time Delay</u>

$$G_p(s) = G(s)\, e^{-s\theta} \quad , \quad 0 \leq \theta \leq \theta_{max}$$

We then have:

$$r_i = \sup_{0 \leq \theta \leq \theta_{max}} \left\{ |G(j\omega_i)(e^{j\omega_i \theta} - 1)| \right\} \quad , \forall i$$

Thus: $W_M(s)$ must be designed such that

$$|W_M(j\omega_i)| \geq |e^{j\omega_i \theta} - 1| \quad , \forall i$$

$W_m(s)$ can be taken as $W_m(s) = \dfrac{\theta_{max}\, s}{\frac{\theta_{max}}{2} s + 1}$