Functional Controllability (not to be confused w/ state controllability)

Consider the system: $G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+2} \\ \frac{2}{s+1} & \frac{4}{s+2} \end{bmatrix}$ (2 inputs, 2 outputs)

The second row is 2x the first row, so obviously (with zero initial state)

$$Y_2(t) = 2y_1(t)$$

regardless of the inputs. This system is not functionally controllable.

Consider another system $G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+2} & \frac{4}{s+2} \end{bmatrix}$

The outputs are not linearly dependent anymore, but the inputs are. That is, the effect of the second input is exactly twice of the first one. So it is as if one input is driving two outputs.

$G(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{2}{s+2} \end{bmatrix} \rightarrow Y_2(s) = Y_1(s)\,\frac{2(s+1)}{(s+2)}$ ← Y_1 and Y_2 are dynamically coupled

This system is not functionally controllable.

Normal rank: the rank of the transfer matrix $G(s)$ at all $s \in \mathbb{C}$ except possibly for a finite number of points (the zeros).

Functional controllability: If $G(s)$ is $l \times m$, it is functionally controllable if the normal rank is l.

Ex: $G(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{2}{s+2} \end{bmatrix}$ is not functionally controllable, but

$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+2} \end{bmatrix}$ is.

Note that this is an input-output notion, and thus is not to be confused with state-controllability (which is a realization property)

For example: $\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$, $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$

is a realization for $G(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{2}{s+2} \end{bmatrix}$. It can be checked that it's state controllable.

Another example: double inverted pendulum

Functional controllability of state-space representation:

Suppose that $\dot{x} = Ax + Bu$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

$y = Cx$, $y \in \mathbb{R}^l$

is a minimal representation (contr & obs)

$Y(s) = C(sI - A)^{-1} B\, U(s)$. The normal rank of $(sI - A)^{-1}$ is n

The system is not functionally controllable if:

- rank $C < l$ (coupled outputs)
- rank $B < l$ (insufficient inputs)
- $n < l$ (same as rank $C < l$)

MIMO time delay

MIMO time delay also imposes performance limitation (like SISO).

If θ_{ij} is the time delay of $G_{ij}(s)$, the time delay to the i-th output can be bounded by:

$$\theta_i^{min} = \min_j \theta_{ij}$$

Interpretation: it is the smallest delay possible from any input to output i.

However, time delay can help in MIMO systems. It can decouple the outputs so as to achieve functional controllability.

Example: $G_1(s) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not functionally controllable, but

$G_2(s) = \begin{bmatrix} 1 & 1 \\ e^{-s\theta} & 1 \end{bmatrix}$ is. The determinant of $G_2(s) = 1 - e^{-s\theta}$

which is nonzero unless $s = \frac{j 2k\pi}{\theta}$, $k = 0, 1, 2, \ldots$

Back to Bound on peaks of S & T

If $G_\theta(s) = \theta(s)G(s)$ is the plant transfer matrix, with

$$\theta(s) = \begin{bmatrix} e^{-\theta_1 s} & & 0 \\ & e^{-\theta_2 s} & \\ 0 & & \ddots \; e^{-\theta_\ell s} \end{bmatrix}$$

and $G(s)$ has n_z RHP zeros at $s = z_i, i = 1 \ldots n_z$ with output directions $y_{z,i}$

and n_p RHP poles at $s = p_i, i = 1 \ldots n_p$ with output directions $y_{p,i}$

Assume that p_i and z_i are all distinct, then

$\|T\|_\infty \geq \Lambda$, where Λ is the largest eigenvalue of

$$\left(Q_\theta^{-1/2}\left(Q_p + Q_{zp} Q_z^{-1} Q_{zp}^H\right) Q_\theta^{-1/2}\right)$$

$$[Q_\theta]_{ij} = \frac{y_{p,i}^H \theta(\bar{p}_i)\theta(p_j) y_{p,j}}{\bar{p}_i + p_j}$$

(Q_p, Q_z, and Q_{zp} were introduced in Lecture 18)

$$\|S\|_\infty \geq \Lambda - 1$$

Special case: only one RHP pole and zero, then:

$$\Lambda = \frac{1}{\|\theta(p) y_p\|_2} \sqrt{\sin^2\phi + \left|\frac{z+p}{z-p}\right|^2 \cos^2\phi}$$

where ϕ is the angle between y_z and y_p.

Interpretation: alignment between y_p and y_z is important, but also the alignment between the delay elements and y_p.

Bandwidth limitation due to RHP zeros for MIMO systems

For MIMO systems with open loop RHP zero at $s = z$, we also have

$\|W_p S\|_\infty \geq |W_p(z)|$, where W_p is a scalar weight.

Thus if we want $\|W_p S\|_\infty \leq 1$, we need $|W_p(z)| \leq 1$. This will have bandwidth limitation, as discussed for SISO systems

Limitations due to RHP poles

Just like the SISO case, RHP poles also impose performance limitation for MIMO system:

1. Limitation on input usage:

$$\|KS\|_\infty \geq |G_s^{-1}(p)|, \text{ for any RHP pole } p$$

G_s is the stable version of G, i.e. with the unstable poles mirrored to the open LHP.

2. Minimum bandwidth limitation: to stabilize the plant, we need a bandwidth of at least $2|p|$. Bandwidth limitation can also be derived from

$$\|W_T T\|_\infty \geq |W_T(p)| \quad \text{for any RHP pole } p.$$

That is: $|W_T(p)| \leq 1$