Back to SISO Performance Limitations

Some factors that can limit performance:

- time delay
- RHP zero
- RHP pole
- phase lag

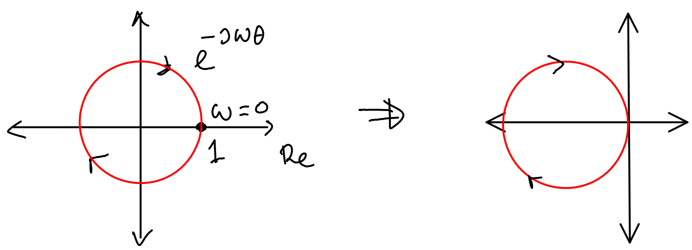
Time Delay

Physically cannot be inverted. The most ideal case:

$$T = e^{-s\theta}$$

This implies $S = 1 - T = 1 - e^{-s\theta}$

$$S(j\omega) = 1 - e^{-j\omega\theta}$$



The peak of $|S|$ is 2

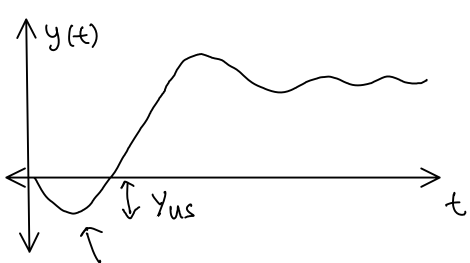
At $\omega = \frac{\pi}{2\theta}$, $|S| = \sqrt{2} \approx 6\text{dB}$

The frequency where $|S|$ crosses 1 for the first time can be calculated at $\approx \frac{1.05}{\theta}$. Thus the closed loop bandwidth w.r.t S is limited by

$$\omega_c < \frac{1}{\theta}$$

RHP zeros

RHP zeros are associated with undershoots in the response



undershoot

For stable SISO plant with n_z real RHP zeros, the output to a step reference crosses zero at least n_z times

Lower bound on y_{us} for systems with one real RHP zero at z:

$$|y_{us}| \geq |y_f| \frac{0.95}{e^{z t_s} - 1}$$

Thus: - The faster the response, the worse the undershoot
- The closer z to 0, the worse the undershoot

High gain instability:

Recall root locust plot: The plot of the closed loop poles as the loop gain varies

$$S(s) = \frac{1}{1 + K L(s)} \rightarrow \text{CL poles } \quad 1 + K \cdot L(s) = 0$$

For $K << 1$, if p is a CL pole then $L(p) \approx \infty$.

For $K >> 1$, if p is a CL pole then $L(p) \approx 0$

Thus as we increase gain, the CL poles tend towards OL zeros

Bandwidth limitation

Recall the bound on peak: $\|W_p S\|_\infty \geq W_p(z)$ for any RHP zero z

For H_∞ synthesis, we want $\|W_p S\|_\infty \leq 1$, therefore we must at least require that $|W_p(z)| \leq 1$

This has a consequence on the design of W_p.

If $W_p(s) = \dfrac{\frac{s}{M} + \omega_B}{s + \omega_B A}$, then

$$\left|\frac{z}{M} + \omega_B\right| \leq \left|z + \omega_B A\right|$$

$\boxed{\text{If } z \text{ is real:}}$ $\quad \omega_B \leq \dfrac{z(1 - \frac{1}{M})}{1 - A} \leftarrow$ limit on bandwidth

Typically $A \approx 0$, so $\omega_B \leq z(1 - \frac{1}{M})$

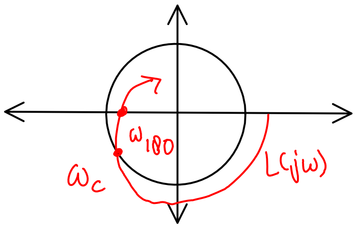
$\boxed{\text{If } z \text{ is complex: } z = x \pm jy}$ For $A \approx 0$,

$$\omega_B \leq -\frac{x}{M} + \sqrt{x^2 + y^2\left(1 - \frac{1}{M^2}\right)}$$

Phase Lag

If $G(s)$ is stable and does not have RHP zeros, but has a lot of phase lag. Eq: $G(s) = K \prod_{i=1}^{N} \frac{1}{(1+s\tau_i)}$, $N \geq 3$

This means that the phase response tends to $-N \cdot 90^\circ$ as $\omega \to \infty$



For stability, we need $\omega_c < \omega_{180}$
However ω_{180} is a contribution from both G and K, Define ω_u such that $\angle G(j\omega_u) = -180^\circ$

For P or PI controller: $\omega_{180} \leq \omega_u$, thus we need $\omega_c < \omega_u$

For PID controller the same bound still (practically) applies as the controllers zero is neglected.

RHP poles

Limitation on input usage: We can derive a bound on the peak of $\|KS\|_\infty$:

$$\|KS\|_\infty \geq |G_s^{-1}(p)| \text{, where } G_s \text{ is the stable version of } G$$

Example: $G(s) = \frac{s+1}{s-2} \to G = G_s \cdot G_c$

$G_c(s) = \frac{s+2}{s-2}$, $G_s(s) = \frac{s+1}{s+2}$

One RHP pole at $p=2$, thus $\|KS\|_\infty \geq |G_s^{-1}(p)| \geq \left| \frac{2+2}{2+1} \right| \geq \frac{4}{3}$

Limitation on lower bandwidth: to stabilize the plant we need fast reaction nd we need large enough CL bandwidth.
CL bandwidth needs to be at least:

- $2p$ for one real RHP pole at p
- $0.67(x + \sqrt{4x^2+3y^2})$ for a pair of RHP poles at $p = x \pm jy$
- $1.15|p|$ for a pair of pure imaginary RHP poles at $\pm j|p|$

RHP pole necessarily implies overshoot in step response. Define an alternative rise time t_r as:

$$t_r \triangleq \max \left\{ \tau \mid y(t) \le \frac{t}{\tau}, \forall t \ge 0 \right\}$$

Then the amount of overshoot can be bounded by:

$$y_{os} \ge 1 + y_f \cdot \frac{p \cdot t_r}{2}$$

For a system with a single RHP pole at p. Here $y_f = \lim_{t \to \infty} y(t)$