Relative Gain Array

Given a transfer matrix of MIMO system with m inputs - m outputs

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1m}(s) \\ \vdots & & & \\ G_{m1}(s) & G_{m2}(s) & \cdots & G_{mm}(s) \end{bmatrix}, \quad G(s) \text{ is nonsingular}$$

The relative gain array of $G(s)$:

$RGA(G) = \Lambda(G) = G \times (G^{-1})^T$ → "$\times$" denotes element wise multiplication

Example: 2x2 system

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \rightarrow G^{-1}(s) = \frac{1}{G_{11}G_{22} - G_{12}G_{21}} \begin{bmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11} \end{bmatrix}$$

$$\Lambda(G) = \begin{bmatrix} \dfrac{G_{11}G_{22}}{G_{11}G_{22} - G_{12}G_{21}} & \dfrac{-G_{12}G_{21}}{G_{11}G_{22} - G_{12}G_{21}} \\[2ex] \dfrac{-G_{12}G_{21}}{G_{11}G_{22} - G_{12}G_{21}} & \dfrac{G_{11}G_{22}}{G_{11}G_{22} - G_{12}G_{21}} \end{bmatrix}$$

Interpretation: Consider pure gain transfer matrix

$$y = G u \rightarrow \Delta y = G \Delta u$$

$\Lambda(G)_{ij} = G_{ij} \cdot (G^{-1})_{ji}$

G_{ij} = gain from input j to output i

= the influence of input j to output i if other inputs are 0

We can coordinate the inputs $u_{i \ldots m}$ such that only one output is changed but the others remain constant

$$\Delta y = G \cdot \Delta u \implies \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = G \Delta u \rightarrow \Delta u = G^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{output } i$$

output i

$\Delta u =$ the i-th column of G^{-1}

$$\frac{\Delta y_i}{\Delta u_j} = \frac{1}{(G^{-1})_{ji}}$$

Thus: $$\Lambda(G)_{ij} = \frac{\left(\frac{\Delta y}{\Delta u}\right)\text{openloop}}{\left(\frac{\Delta y}{\Delta u}\right)\text{close loop}}$$

Consider the 2x2 system: Suppose that we want to control y_1 using u_1

Openloop : $u_2 = 0 \to y_1(s) = G_{11}(s)\, u_1(s)$

Close loop : $y_2 = 0 \to Y_2(s) = G_{21}(s)\, U_1(s) + G_{22}(s)\, U_2(s) = 0$

$$U_2(s) = -\frac{G_{21}}{G_{22}}\, U_1(s)$$

$$\begin{aligned} Y_1(s) &= G_{11}(s)\, U_1(s) + G_{12}(s)\, U_2(s) \\ &= G_{11}(s)\, U_1(s) - \frac{G_{12}G_{21}}{G_{22}}\, U_1(s) \\ &= \frac{G_{11}G_{22} - G_{12}G_{21}}{G_{22}}\, U_1(s) \end{aligned}$$

$$\frac{\text{open loop TF}}{\text{Close loop TF}} = \frac{G_{11}G_{22}}{G_{11}G_{22} - G_{12}G_{21}}$$

Intuitively:

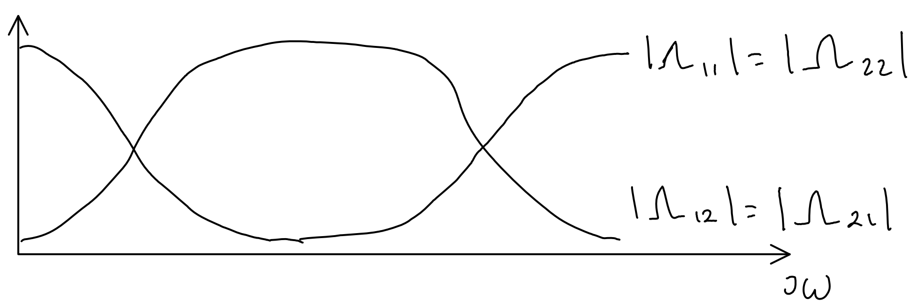
- If we want to control output i with input i, $\forall i = 1, 2, \dots m$ then ideally $\Lambda(G) = I$. For example: in the 2x2 case, $G_{12} = 0$ or $G_{21} = 0$
- If $\Lambda(G)$ is not diagonal dominant, then it suggests input-output pairing permutation

Frequency Dependent RGA: since $\Lambda(G)$ is frequency dependent, different pairing might be suitable for different frequency ranges

Example (Ex 3.11 p. 86):

$$G(s) = \frac{0.01 e^{-5s}}{(s+1.72\cdot 10^{-4})(4.32s+1)} \begin{bmatrix} -34.54(s+0.0572) & 1.913 \\ -30.22s & -9.188(s+6.95\cdot 10^{-4}) \end{bmatrix}$$

$$\Lambda(G) = \frac{1}{34.54\cdot 9.188(s+0.0572)(s+6.95\cdot 10^{-4}) + 1.913\cdot 30.22s} \cdot \begin{bmatrix} 34.54\cdot 9.188(s+0.0572)(s+6.95\cdot 10^{-4}) & 1.913\cdot 30.22s \\ 1.913\cdot 30.22s & 34.54\cdot 9.188(s+0.0572)(s+6.95\cdot 10^{-4}) \end{bmatrix}$$



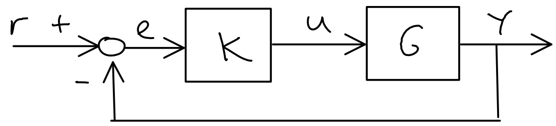
At very low or high frequency, diagonal pairing is best
At mid frequency range, off-diagonal pairing is best

For $s=0$ (steady state RGA): Avoid pairing involving negative RGA element as it might cause instability

Control of Multivariable Plants

* Decentralized control: The controller is diagonal. Three designed techniques: Fully coordinated, sequential design, independent design

- 2-step compensator design approach



Design: $\rightarrow \boxed{K} \rightarrow \; = \; \rightarrow \boxed{K_s} \rightarrow \boxed{W_s} \rightarrow$, and

$W_s G$ is (almost) diagonal, afterwards K_s is designed for the decoupled plant: $G_s = G W_s$

Example: If $G(s)$ does not have RHP zero and bi-proper:

$$G(s) = \frac{1}{s+1} \begin{bmatrix} s+2 & s+3 \\ s & s+2 \end{bmatrix}.$$

Take $W_s = G^{-1}(s) = \dfrac{(s+1)}{(s+2)^2 - s(s+3)} \begin{bmatrix} s+2 & -(s+3) \\ -s & s+2 \end{bmatrix}$, we can then assume

$$K_s(s) = \begin{pmatrix} K_1(s) & 0 \\ 0 & K_2(s) \end{pmatrix},$$ where $K_1(s)$ and $K_2(s)$ are designed separately for unit constant gain plant.

Alternatively: Decouple at frequency range of interest