Input - Output Decoupling Problem (IODP)

Consider the system:

$$\dot{x} = Ax + Bu \qquad \Big\} \; x \in \mathbb{R}^n, \; u \in \mathbb{R}^m, \; y \in \mathbb{R}^\ell$$
$$y = Cx$$

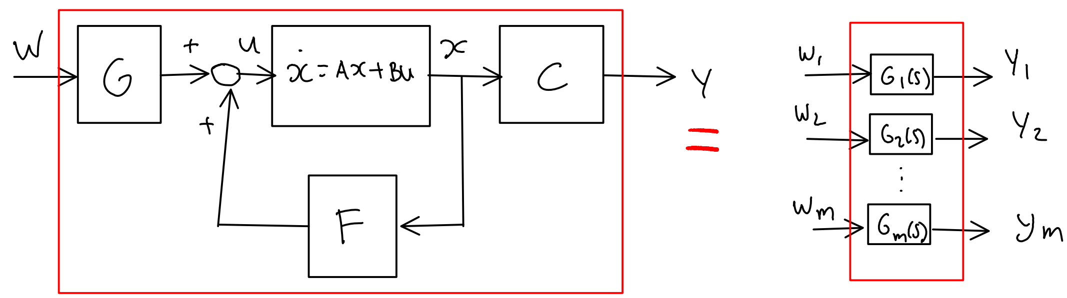
Problem: Design

$$u = Fx + GW, \quad F \in \mathbb{R}^{m \times n}, \quad G \in \mathbb{R}^{m \times m} \text{ invertible}$$

such that the TF from w_i to y_j is 0 if $i \neq j$

Closed Loop system:

$$\dot{x} = (A + BF)x + BGW$$
$$y = Cx$$



Relation to DDP: For each output y_i, $w_j, j \neq i$ is treated as disturbance and rejected.

Detour: Relative Degree

Idea: Consider a state space representation of a SISO system:

$$\boxed{\frac{Y(s)}{U(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_0}{s^n + b_{n-1} s^{n-1} + \cdots + b_0}} \Rightarrow \boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

$\|$ equal

$$\boxed{C(sI - A)^{-1}B + D = G(s)}$$

$$\boxed{\text{The relative degree of this TF is } (n-m)}$$

- If the relative degree is 0, then $n=m$

$$a_m = \lim_{s\to\infty} G(s) = D$$

- If the relative degree >0, then $D = \lim_{s\to\infty} G(s) = 0$

Useful fact: $(sI-A)^{-1} = \frac{1}{s}\left(I - \frac{A}{s}\right)^{-1}$

If s is large enough such that $\left\|\frac{A}{s}\right\| < 1$, then

$$I + \frac{A}{s} + \frac{A^2}{s^2} + \cdots = \left(I - \frac{A}{s}\right)^{-1} \quad \text{(proof: just multiply them)}$$

Thus: $(sI-A)^{-1} = \frac{1}{s}\left(I + \frac{A}{s} + \frac{A^2}{s^2} + \cdots\right)$

- If the relative degree is 1,

$$a_m = \lim_{s\to\infty} sG(s) = \lim_{s\to\infty} C\left(I - \frac{A}{s}\right)^{-1} B = CB \neq 0$$

- If the relative degree is 2, $CB = 0$

$$a_m = \lim_{s\to\infty} s^2 G(s) = \lim_{s\to\infty} C\, s^2 \cdot \left(\frac{1}{s}\cdot\left(I + \frac{A}{s} + \cdots\right)\right) B$$

$$= \lim_{s\to\infty} C\left(sI + A + \frac{A^2}{s} + \cdots\right) B$$

$$= CAB \neq 0$$

$\vdots$

- If the relative degree is $m<n$, $CB=0$, $CAB=0$, $\cdots$, $CA^{m-2}B = 0$

$$a_m = \lim_{s\to\infty} s^m G(s) = \lim_{s\to\infty} C s^{m-1}\left(I + \frac{A}{s} + \cdots\right) B$$

$$= CA^{m-1}B \neq 0$$

- If $CA^j B = 0$, for all $j = 0, 1, 2, \ldots, n-1$, then $G(s) \equiv 0$

Back to IODP

$\boxed{\text{The Solution :}}$

Denote the i-th row of C as C_i

Define $d_i = \min\{j \mid C_i A^j B \neq 0, j = 0, 1, \ldots, n-1\}$, or
if $C_i A^j B = 0$, for $j = 0, 1, \ldots, n-1$, then $d_i = n-1$

Interpretation : d_i is the smallest relative degree to i-th output $-$ 1

Observe that if $k \leq d_i$:

$$C_i (A+BF)^0 = C_i$$
$$C_i (A+BF) = C_i A + C_i BF = C_i A$$
$$C_i (A+BF)^2 = C_i A (A+BF) = C_i A^2$$
$$\vdots$$
$$C_i (A+BF)^k = C_i A^k$$
$$C_i (A+BF)^{d_i+1} = C_i A^{d_i} (A+BF)$$
$$\vdots$$
$$C_i (A+BF)^{n-1} = C_i A^{d_i} (A+BF)^{n-1-d_i}$$

Then, observe that for the close loop system:

$$y_i = C_i x$$
$$\frac{dy_i}{dt} = C_i \dot{x} = C_i[(A+BF)x + BGw] = C_i(A+BF)x = C_i A x$$
$$\frac{d^2 y_i}{dt^2} = C_i A \dot{x} = C_i A[(A+BF)x + BGw] = C_i (A+BF)^2 x = C_i A^2 x$$
$$\vdots$$
$$y_i^{(d_i)} \equiv \frac{d^{d_i} y_i}{dt^{d_i}} = C_i (A+BF)^{d_i} x$$

$$y_i^{(d_i+1)} \equiv \frac{d^{(d_i+1)} y_i}{dt^{(d_i+1)}} = C_i (A+BF)^{d_i} [(A+BF)x + BGw]$$
$$= C_i (A+BF)^{d_i+1} x + C(A+BF)^{d_i} BGw$$
$$= C_i A^{d_i} (A+BF) x + C A^{d_i} BGw$$

We stack together:

$$y_1^{(d_1+1)} = \left(C_1 A^{d_1+1} + C_1 A^{d_1} BF\right)x + C_1 A^{d_1} BGW$$

$$y_2^{(d_2+1)} = \left(C_2 A^{d_2+1} + C_2 A^{d_2} BF\right)x + C_2 A^{d_2} BG\,W$$

$$\vdots$$

$$y_m^{(d_m+1)} = \left(C_m A^{d_m+1} + C_m A^{d_m} BF\right)x + C_m A^{d_m} BGW$$

Define: $P \equiv \begin{bmatrix} C_1 A^{d_1+1} \\ C_2 A^{d_2+1} \\ \vdots \\ C_m A^{d_m+1} \end{bmatrix}$; $Q \equiv \begin{bmatrix} C_1 A^{d_1} B \\ C_2 A^{d_2} B \\ \vdots \\ C_m A^{d_m} B \end{bmatrix}$

$$\begin{bmatrix} y_1^{(d_1+1)} \\ y_2^{(d_2+1)} \\ \vdots \\ y_m^{(d_m+1)} \end{bmatrix} = (P + QF)x + QGW$$

Theorem: The Input Output Decoupling Problem is solvable if and only if Q is nonsingular

Construction: Take $F = -Q^{-1}P$

$G = Q^{-1}$

Beware: Might be internally unstable, i.e. $(A+BF)$ might not be Hurwitz

Example: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$; $C^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \Rightarrow C_1 B = \begin{bmatrix} 1 & 1 \end{bmatrix}$; $d_1 = 0$

$C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \Rightarrow C_2 B = \begin{bmatrix} 1 & 0 \end{bmatrix}$; $d_2 = 0$

$$Q = \begin{bmatrix} C_1 B \\ C_2 B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \text{invertible} \rightarrow \text{IODP is solvable}$$

$$P = \begin{bmatrix} C_1 A \\ C_2 A \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}; \quad F = -Q^{-1}P = -\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$G = Q^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \text{Close loop TF: } \begin{aligned} \dot{y}_1 &= w_1 \\ \dot{y}_2 &= w_2 \end{aligned}$$

From the TF perspective:

$$y_1 = x_1$$
$$\dot{y}_1 = x_2 + u_1 + u_2$$
$$\ddot{y}_1 = \dot{x}_2 + \dot{u}_1 + \dot{u}_2 = -x_2 + \dot{u}_1 + \dot{u}_2 = u_1 + u_2 - \dot{y}_1 + \dot{u}_1 + \dot{u}_2$$
$$\ddot{y}_1 - \dot{y}_1 = u_1 + \dot{u}_1 + u_2 + \dot{u}_2$$
$$Y_1(s) = \begin{bmatrix} \frac{s+1}{s^2-1} & \frac{s+1}{s^2-1} \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$y_2 = x_3$$
$$\dot{y}_2 = -x_3 + u_1 = -y_2 + u_1 \rightarrow Y_2(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s^2-s} & \frac{s+1}{s^2-s} \\ \frac{1}{s+1} & 0 \end{bmatrix} \begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix} \rightarrow Q = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$