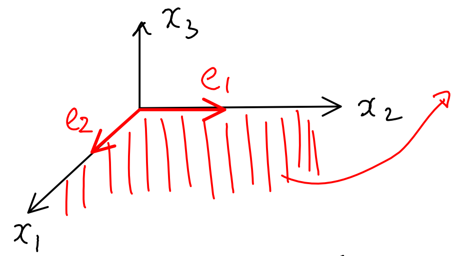
Numeric Implementation

Consider $\mathbb{R}^n$ as the underlying space.

How to represent a subspace $\mathcal{V} \subset \mathbb{R}^n$? Suppose $\dim \mathcal{V} = p \leq n$.

$\boxed{\text{Image representation:}}$ As the image of a matrix of its basis vector

Example: $n=3$



$$\operatorname{im}\left[e_1 \,\vdots\, e_2\right] = \operatorname{im}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\mathcal{V} = \operatorname{Im} M_V \rightarrow M_V = [e_1 \,\vdots\, e_2 \,\vdots\, \cdots \,\vdots\, e_p]$

M_V is a minimal representation of $\mathcal{V}$ if it has full column rank ($\operatorname{rank} M_V = p$)

Otherwise, reduce the number of columns through elementary column operations.

Example: $M_V = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ subtract the first and second columns from the third

$$M_V = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightarrow \operatorname{rank} = 2$, thus $\dim \mathcal{V} = 2$

with MATLAB, a quick way to obtain minimal representation is through singular value decomposition:

$$M_V = U \Sigma V^T$$

The rank of M_V is # nonzero elements in Σ, a minimal representation can be obtain by selecting the columns of U corresponding to the nonzero elements in Σ.

Eg: $M_V = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$

Thus a minimal representation of $\mathcal{V} = \text{im}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\boxed{\text{Kernel Representation}}$: Represent $\mathcal{V}$ as the kernel of a matrix $N_\mathcal{V}^T$

$$\mathcal{V} = \ker N_\mathcal{V}^T \rightarrow x \in \mathcal{V} \Leftrightarrow N_\mathcal{V}^T x = 0$$

Geometric interpretation: the columns of $\mathcal{V}$ span the orthogonal space complement to $\mathcal{V}$

Ex: $\mathcal{V} = \ker \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

As before, the kernel representation is minimal if N has full column rank.

Rank $N_\mathcal{V} = n - p$

Conversion:

* Given $M_V = U \Sigma V^T$,
  $N_\mathcal{V}$ can be taken as the columns of U corresponding to the zero diagonal entries of Σ

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Space summation: Given $\mathcal{V} = \text{im}\, M_\mathcal{V}$ and $\mathcal{W} = \text{im}\, M_\mathcal{W}$, both minimal

$$\mathcal{V} + \mathcal{W} = \text{im}\,[M_\mathcal{V} \vdots M_\mathcal{W}] \leftarrow \text{not necessarily minimal}$$

Intersection: Given $\mathcal{V} = \ker N_\mathcal{V}^T$ and $\mathcal{W} = \ker N_\mathcal{W}^T$, both minimal

$$\mathcal{V} \cap \mathcal{W} = \ker\,[N_\mathcal{V} \vdots N_\mathcal{W}]^T \leftarrow \text{not necessarily minimal}$$

Inclusion: Given two subspaces $\mathcal{V} \subset \mathbb{R}^n$ and $\mathcal{W} \subset \mathbb{R}^n$

Suppose that $\dim \mathcal{V} \leq \dim \mathcal{W}$. How to check if $\mathcal{V} \subset \mathcal{W}$?

$\mathcal{V} \subset \mathcal{W}$ iff $\dim(\mathcal{V} + \mathcal{W}) = \dim \mathcal{W}$ $\leftarrow$ image rep

$\mathcal{V} \subset \mathcal{W}$ iff $\dim(\mathcal{V} \cap \mathcal{W}) = \dim \mathcal{V}$ $\leftarrow$ kernel rep

<u>Inverse Map</u> : Given $\mathcal{V} = \operatorname{im} M_V \subset \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$. How to represent $A^{-1}\mathcal{V}$

- First, compute $\mathcal{W} = \ker A$
- Obtain an image representation of $\mathcal{W}$: $\mathcal{W} = \operatorname{im} M_W$
- Next, compute $\mathcal{Z} = \mathcal{V} \cap \operatorname{im} A$
- obtain an image representation of $\mathcal{Z}$: $\mathcal{Z} = \operatorname{im} M_Z$
- We compute M_Q by solving $A M_Q = M_Z$, which is guaranteed to have solution since $\mathcal{Z} \subset \operatorname{im} A$,
- Define $\mathcal{Q} = \operatorname{im} M_Q$,
- $A^{-1}\mathcal{V} = \mathcal{W} + \mathcal{Q}$

- Note : if A is invertible, then $\mathcal{W} = \{0\}$, $\mathcal{Z} = \mathcal{V}$, and thus $A^{-1}\mathcal{V} = \operatorname{im}(A^{-1} M_V)$

Example : Given $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Find the largest controlled invariant subspace of $\ker C$

Iteration :

$V_0 = \ker C$

$V_1 = V_0 \cap A^{-1}(V_0 + \operatorname{im} B)$

$$V_0 + \operatorname{im} B = \operatorname{im} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = \operatorname{im} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{matrix}$$

$$A^{-1}(V_0 + \text{im}\, B) = \text{im} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$V_1 = V_0 \cap A^{-1}(V_0 + \text{im}\, B) = \text{im} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_2 = V_1 \cap A^{-1}(V_1 + \text{im}\, B) \rightarrow V_1 + \text{im}\, B = \text{im} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{-1}(V_1 + \text{im}\, B) = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$V_2 = V_1 \cap A^{-1}(V_1 + \text{im}\, B) = \text{im} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$V_3 = V_2 \cap A^{-1}(V_2 + \text{im}\, B) \Rightarrow V_2 + \text{im}\, B = \text{im} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = V_1 + \text{im}\, B$$

$$A^{-1}(V_2 + \text{im}\, B) = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow V_3 = V_2 = V^*$$

DDP with two-degrees of freedom

Given : $\dot{x} = Ax + Bu + Gd$

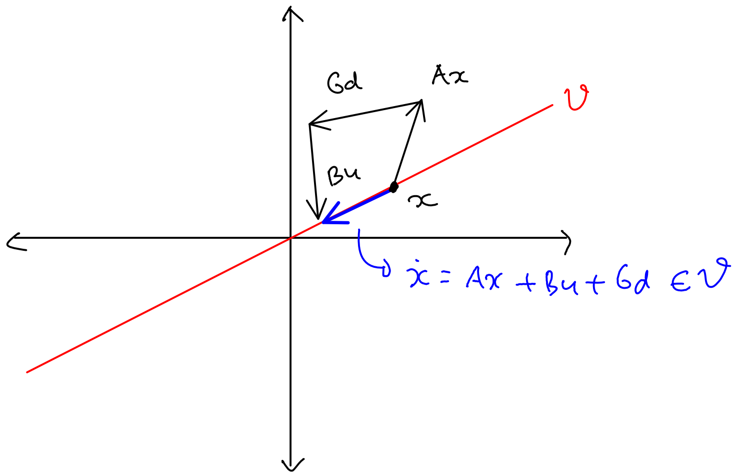
$y = Cx$

Suppose that we want to decouple d from y by using 2 DOF control :

$$u = -Kx + Hd$$

Geometrically, this means finding a subspace $\mathcal{V} \subset \ker C$ such that

$$A\mathcal{V} + \operatorname{im} G \subset \mathcal{V} + \operatorname{im} B$$



Theorem: DDP with 2DOF is solvable iff $\operatorname{im} G \subset \mathcal{V}^* + \operatorname{im} B$

Proof: (If) Suppose that $\operatorname{im} G \subset \mathcal{V}^* + \operatorname{im} B$, then $\exists H_1, H_2$ such that $\forall d \in \mathbb{R}^d$ such that : $Gd = H_1 d + BH_2 d$, where $H_1 d \in \mathcal{V}^*$

$H_2 d \in \mathbb{R}^m$

Thus we can use $H = -H_2$, and K is the state feedback that makes $\mathcal{V}^*$ invariant.

$$\dot{x} = Ax + Bu + Gd = (A - BK)x + \underbrace{Gd - BH_2 d}_{\in \mathcal{V}^*}$$

(only if) Suppose that $u = -Kx + Hd$ solves the DDP with 2DOF,

Define $\tilde{G} = G + BH$. We know that $\operatorname{im}\tilde{G} \subset \mathcal{V}^*$ from the DDP.

However $\operatorname{im} G \subset \operatorname{im} \tilde{G} + \operatorname{im} B \subset \mathcal{V}^* + \operatorname{im} B$