ECSE 6420: Nonlinear Control Systems

Midterm Exam. Due date: 5 March 2009

Exam rules: You are not allowed to collaborate with each other. The instructor and the TA will not provide any further hint on how to solve the problems. If you have any question about understanding the statement of the problems, or if you think there's an error in the problems, send an email to the instructor.

Points: Problem 1 = 3+3+5+3+3+3+5+15 pts, Problem 2 = 5+15 pts, Problem 3 = 10+5+10+5 pts, Problem 4 = 10 pts

Problem 1. The celebrated design of a genetic toggle switch (see Nature, 403(6767), pp. 339-42, 2000) can be modeled as a planar dynamical system

$$\dot{x} = f(y) - x,$$
$$\dot{y} = f(x) - y,$$

where the function $f(\cdot)$ is defined as

$$f(x) = \begin{cases} 1, & x \leq \frac{1}{4}, \\ \frac{3}{2} - 2x, & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 0, & x \geq \frac{3}{4}. \end{cases}$$

(a) Plot and show that the function $f(x)$ is continuous.
(b) Plot the nullclines of the system. (That is, the curves corresponding to $\dot{x} = 0$ and $\dot{y} = 0$, respectively.)
(c) Find the equilibria of the systems and determine their types.
(d) Prove that the positive quadrant $\{(x, y) \mid x \geq 0, y \geq 0\}$ is invariant.
(e) Simulate several trajectories (with different initial conditions) of the system and plot them on the xy plane. Try to spread the initial conditions in the square that is given by $0 \leq x, y \leq 1$.
(f) Show that the vector field is symmetric with respect to the line $y = x$.
(g) Show that it is not possible for the trajectory to grow unbounded, that is,

$$\lim_{t \to \infty} \left\| \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \right\| < +\infty.$$

Hint: Try to compute the time-derivative of $V(x, y) = x^2 + y^2$ and use the shape of $f(\cdot)$.
(h) Using, among others, the results from (f) and (g), determine the regions of attraction of the two stable equilibria in the positive quadrant.

Problem 2. The following system is used to model oscillations in biochemical interactions:

$$\dot{x}_1 = k \cdot x_2 \left(1 - \frac{x_1}{1 + x_2^2}\right)$$
$$\dot{x}_2 = 10 - x_2 - \frac{4x_1 x_2}{1 + x_2^2}$$

where x_1 and x_2 represent the concentrations of two biochemical components; and $k > 0$ is a positive constant.
(a) Find the equilibrium of this system.
(b) **Derive** a sufficient condition for k, under which the system is guaranteed to have a periodic orbit inside the region

$$S = \{\ (x_1, x_2) \mid x_1 \geq 0,\ x_2 \geq 0,\ x_1 \leq 101,\ x_2 \leq 10\}.$$

Problem 3.(Krasovkii's method) Consider the system $\dot{x} = f(x)$ with $f(0) = 0$, $x \in \mathbb{R}^n$. Assume that $f(x)$ is continuously differentiable and its Jacobian $J(x) := \frac{\partial f}{\partial x}(x)$ satisfies

$$PJ(x) + J^T(x)P \leq -I$$

for all $x \in \mathbb{R}^n$, where P is a symmetric positive definite matrix.
(a) Show that $f(x) = \int_0^1 J(\sigma x)x\ d\sigma$. Remember that x is a vector!
(b) Using (a), show that

$$x^T P f(x) + f^T(x) P x \leq -x^T x.$$

(c) Show that $V(x) = f^T(x) P f(x)$ is positive definite for all $x \in \mathbb{R}^n$ and radially unbounded.
(d) Show that the origin is globally asymptotically stable.

Problem 4. Consider the system

$$\dot{x} = -a(I_n + S(x) + xx^T)x,$$

where a is a positive constant, I_n is the $n \times n$ identity matrix, and $S(x)$ is an x-dependent skew symmetric matrix. Show that the origin is globally asymptotically stable.