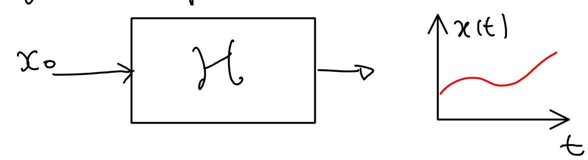
Sensitivity Analysis

Idea: $\dot{x} = f(x, t)$; $x(0) = x_0$

Consider the system as a map from the initial state to the solution trajectory in a finite interval $[0, \tau]$



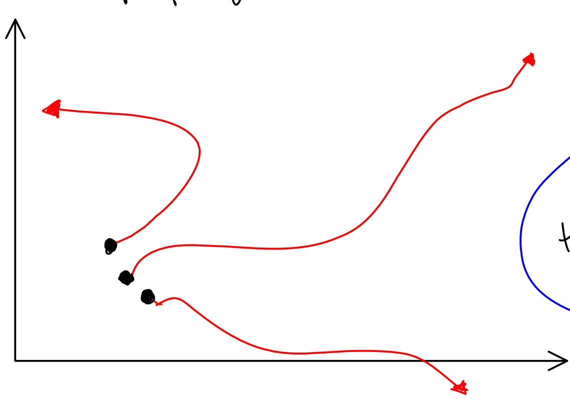
Question: Is $\mathcal{H}$ continuous ?
Does small perturbations in x_0 result in small perturbation in $x(t)$?

Recall that a function $\varphi : X \to Y$, both X and Y are normed spaces, is continuous at $x \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|x' - x\| < \delta \to \|\varphi(x') - \varphi(x)\| < \varepsilon$$

Why is continuity of $\mathcal{H}$ important ?

* limit of simulation
* chaos theory (butterfly effect)
* robustness property



Unrobust property limits the usefulness of simulation see Thm 3.4

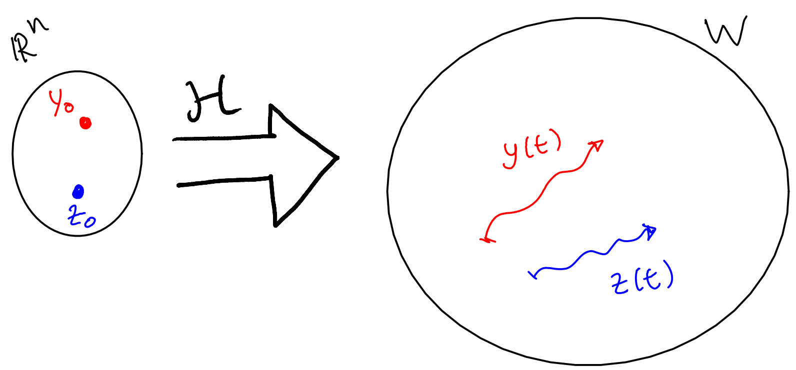
Theorem: Suppose that $f(x,t)$ is piecewise continuous in t, and Lipschitz in x for $(x,t) \in W \times [0,T]$, $W \subset \mathbb{R}^n$ is an open connected set, and L is the Lipschitz constant.

$$\forall t \in [0,T],\ x,y \in W,\quad \|f(x,t) - f(y,t)\| \le L\|x-y\|$$

Suppose that $y(t)$ and $z(t)$ are solutions of:

$$\dot{y}(t) = f(y,t),\ y(0) = y_0$$
$$\dot{z}(t) = f(z,t),\ z(0) = z_0$$

such that $\forall t \in [0,T]$, $\begin{cases} y(t) \in W \\ z(t) \in W \end{cases}$



Then, $\|y(t) - z(t)\| \le \|y_0 - z_0\| e^{Lt}$, $\forall t \in [0,T]$

This establishes the continuity of $\mathcal{H}$! (Why?)

Proof: $y(t) = y_0 + \int_0^t f(y(\tau),\tau)\,d\tau \quad \ldots\ldots (1)$

$z(t) = z_0 + \int_0^t f(z(\tau),\tau)\,d\tau \quad \ldots\ldots (2)$

(1)-(2): $y(t) - z(t) = y_0 - z_0 + \int_0^t f(y(\tau),\tau) - f(z(\tau),\tau)\,d\tau$

$$\|y(t) - z(t)\| \le \|y_0 - z_0\| + \int_0^t L \cdot \|y(\tau) - z(\tau)\|\,d\tau$$

If $x(t) \le a + \int_0^t b\,x(\tau)\,d\tau$ and $b>0$, then $x(t) \le a e^{bt}$

Gronwall-Bellman

Proof: Define $q(t) \triangleq \int_0^t b\,x(\tau)\,d\tau$, thus

$$a + q(t) - x(t) \equiv \Delta(t) \ge 0$$

$$x(t) = a + q(t) - \Delta(t)$$

$$\dot{q}(t) = b\,x(t) = b\cdot a + b q(t) - b\Delta(t)$$

$$\dot{q} - bq = ba - b\Delta(t)$$

$$q(t) = \int_0^t e^{b(t-\tau)}(ba - b\Delta(\tau))\,d\tau$$

$$\le ba \int_0^t e^{b(t-\tau)}\,d\tau = a(e^{bt} - 1)$$

Thus $q(t) \le a e^{bt} - a$; $x(t) \le a + q(t) \le a e^{bt}$

Therefore: $\|y(t) - z(t)\| \le \|y_0 - z_0\|\, e^{Lt} \quad \forall t \in [0,T]$

See Thm 3.4 in the book for a stronger version, i.e. if the dynamics $f(x,t)$ is also perturbed.

Example: Apply the theorem to linear systems:

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

What is the Lipschitz constant?

$$L = \sup \frac{\|Ax\|}{\|x\|}, \quad x \in \mathbb{R}^n$$

$$= \bar{\sigma}(A) \quad \text{(Largest singular value of } A\text{)}$$

Thus: $\|y(t) - z(t)\| \le \|y_0 - z_0\|\, e^{\bar{\sigma}(A)t}$

Suppose that $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$\sigma(A_1) = \sigma(A_2) = 1$, but:

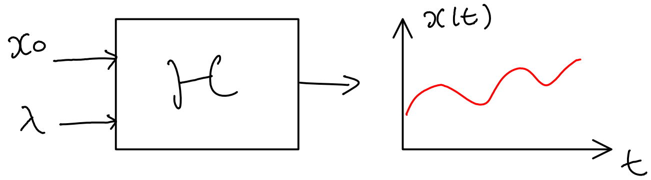
$$\begin{array}{l|l} y_1(t) = y_0 \cdot e^{t} & y_2(t) = y_0 e^{-t} \\ z_1(t) = z_0 \cdot e^{t} & z_2(t) = z_0 e^{-t} \\ \|y_1(t) - z_1(t)\| = \|y_0 - z_0\| e^{t} & \|y_2(t) - z_2(t)\| = \|y_0 - z_0\| e^{-t} \end{array}$$

Lipschitz constant tends to give worst case (conservative) bound.

Parameterized Systems

$$\dot{x} = f(x, t, \lambda), \quad x \in \mathbb{R}^n, \quad \lambda \in \Lambda \subset \mathbb{R}^p, t \in [0, T]$$

λ are the system parameters.



See Thm 3.5 about continuity of H w.r.t x_0 and λ.

Sufficient conditions: 1. $f(x, t, \lambda)$ is Lipschitz in x (uniformly in t, λ)

2. $f(x, t, \lambda)$ is uniformly continuous

Sensitivity w.r.t parameter variation

Consider the system trajectory:

$$x(t) = x_0 + \int_0^t f(x(x_0, \tau, \lambda), \tau, \lambda)\, d\tau$$

Differentiate w.r.t λ:

$$\frac{\partial x}{\partial \lambda}(t) = \int_0^t \left(\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \lambda} + \frac{\partial f}{\partial \lambda} \right) d\tau$$

Denote $\frac{\partial x}{\partial \lambda}(t) \equiv S(t)$

$\hookrightarrow S \in \mathbb{R}^{n \times p}$

Thus: $\boxed{\frac{ds}{dt} = \frac{\partial f}{\partial x} s + \frac{\partial f}{\partial \lambda}}$

Note: $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \lambda}$ are functions of x, t, and λ

Two alternatives for solving the sensitivity equation:

Method 1:

1. Solve $x(t, x_0, \lambda_0)$
2. Compute the Jacobian $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \lambda}$
3. Solve the sensitivity equation as (most probably numerically)

$$\frac{ds}{dt} = A(t) s + B(t)$$

Method 2: Consider S as extra state variables, then solve S together with x.

Example:

$$\dot{x} = \begin{bmatrix} \lambda & 1 \\ -1 & 0 \end{bmatrix} x \quad , \quad x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$, nominal parameter $\lambda_0 = 0$

Thus:

1) We solve for $x(t, x_0, \lambda_0)$ and obtain:

$$x_1(t) = \sin t$$
$$x_2(t) = \cos t$$

2) $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; $\frac{\partial f}{\partial \lambda} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$

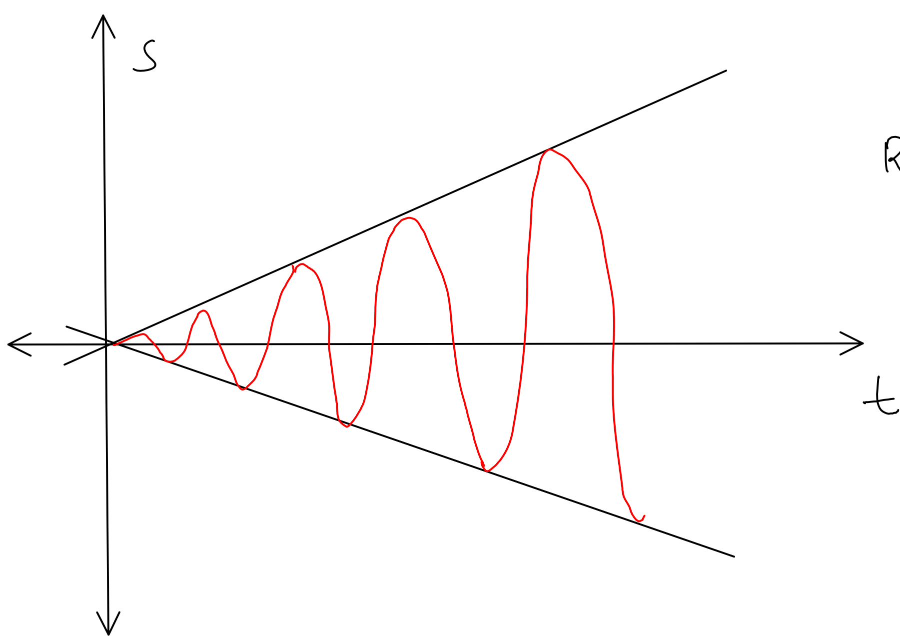
3) Sensitivity equation:

$$\frac{ds}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} s + \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \; ; \; s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\frac{ds}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} s + \begin{bmatrix} \sin t \\ 0 \end{bmatrix} \; ; \; s(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{e^{At} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}}$$

$$s(t) = \int_0^t \begin{bmatrix} \cos(t-\tau)\sin\tau \\ -\sin(t-\tau)\sin\tau \end{bmatrix} d\tau$$



Resonance