Fundamental Properties : Existence and Uniqueness of solution

Normed linear space: A linear space $\mathcal{X}$ equipped with an operator $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a normed linear space if :

1. $\|x\| \geq 0, \ \forall x \in \mathcal{X}$ and $\|x\| = 0$ if and only if $x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|, \ \forall x, y \in \mathcal{X}$ (triangular inequality)
3. $\|\alpha x\| = |\alpha| \, \|x\|, \ \forall \alpha \in \mathbb{R}, x \in \mathcal{X}$

Examples: - $\mathbb{R}^n$ equipped with Euclidean norm

- Space of ~~se~~ real sequences $\{x_i\}_{i=1\ldots\infty}$ with l_1 norm (Prove it!)

$$\|x\| \triangleq \sum_{i=1}^{\infty} |x_i|$$

- space of continuous functions $x : \mathbb{R}_+ \to \mathbb{R}$ with L_2 norm

$$\|x\| = \left[\int_0^{\infty} x^2(t) \, dt \right]^{1/2}$$

Ball : $B(x_0, r) \triangleq \{ x \mid \|x - x_0\| \leq r \}$

Convergence: A sequence $\{x_n\}$ in a normed linear space $\mathcal{X}$ converges to $\bar{x} \in \mathcal{X}$ if

$$\|x_k - \bar{x}\| \to 0 \quad \text{as} \quad k \to \infty$$

Closed set: A subset $S \subset \mathcal{X}$ is closed if and only if every convergent sequence in S has a limit in S.

Cauchy sequence: A sequence $\{x_n\} \in \mathcal{X}$ is a Cauchy sequence if

$$\|x_n - x_m\| \to 0 \quad \text{as} \quad n, m \to \infty$$

Note: convergence $\Rightarrow$ Cauchy

Banach space: is a complete normed space, i.e, every Cauchy sequence converges to element of the space.

Example: Rational numbers are not complete
Real numbers are complete

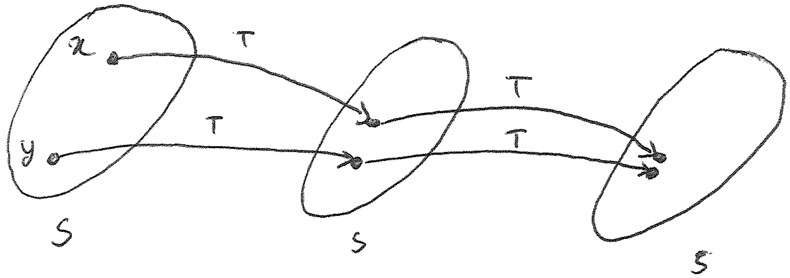
Contraction Mapping Thm: Let S be a closed subset of a Banach space $\mathcal{X}$ and let $T: S \to S$. Suppose that

$$\|T(x) - T(y)\| \leq \rho \|x - y\|, \ \forall x, y \in S \quad \text{for some } 0 \leq \rho < 1$$

then there exists a unique $x^* \in S$ such that $x^* = T(x^*)$.

Note: x^* is called a fixed point of the map T.

Proof:



Take any $x_1 \in S$ and form the sequence $\{x_k\}_{k=1 \ldots \infty}$ by

$$x_{k+1} \triangleq T(x_k), \quad \text{then}$$

$$\|x_{k+1} - x_k\| = \|T(x_k) - T(x_{k-1})\| \leq \rho \|x_k - x_{k-1}\| \leq \rho^{k-1} \|x_2 - x_1\|$$

We'll show that $\{x_k\}$ is a Cauchy sequence:

$$\begin{aligned}
\|x_{k+r} - x_k\| &\leq \|x_{k+r} - x_{k+r-1}\| + \|x_{k+r-1} - x_{k+r-2}\| + \ldots + \|x_{k+1} - x_k\| \\
&\leq \left[\rho^{k+r-2} + \rho^{k+r-3} + \ldots + \rho^{k-1}\right] \|x_2 - x_1\| \\
&\leq \frac{\rho^{k-1}}{1-\rho} \|x_2 - x_1\|
\end{aligned}$$

Since $\frac{\rho^{k-1}}{1-\rho} \to 0$ as $k \to \infty$, the sequence is Cauchy, and since $\mathcal{X}$ is Banach,

$$x_k \to x^* \in \mathcal{X} \quad \text{as } k \to \infty$$

Since S is closed, $x^* \in S$ too.

Show that $x^* = T(x^*)$: Observe that

$$\|x^* - T(x^*)\| \leq \|x^* - x_k\| + \|\overset{\overset{T(x_{k-1})}{\parallel}}{x_k} - T(x^*)\|$$
$$\leq \|x^* - x_k\| + \rho\,(x_{k-1} - x^*)$$

The r.h.s can be made arbitrarily small, thus $\|x^* - T(x^*)\| = 0$, i.e. $x^* = T(x^*)$

<u>Uniqueness</u>. Suppose that $x^* = T(x^*)$, $y^* = T(y^*)$, then $\|x^* - y^*\| \leq \rho \|x^* - y^*\|$

which implies that $\|x^* - y^*\| = 0$, since $\rho < 1$.

Local existence and uniqueness ~~theorem~~

Consider a system: $\dot{x} = f(x,t)$, $t \geq 0$, $x(0) = x_0$ . . . . . $(*)$

A solution of $(*)$ is a continuosly differentiable function of time $x(t)$ satisfying

$$x(t) = x_0 + \int_0^t f(x(\tau),\tau)\,d\tau$$

observe: Fixed point flavor!

Theorem for local existence & uniqueness

Suppose that $f(x,t)$ is continuous in x and t, and that there exist T, r, k, h such that for all $t \in [0,T]$ we have:

$$|f(x,t) - f(y,t)| \leq k|x-y|, \quad \forall x,y \in B(x_0, r)$$

$$|f(x_0,t)| \leq h$$

then $(*)$ has exactly one solution on some time interval $[0,\delta]$.

<u>Proof</u>: Use contraction mapping thm for the space of continuous ~~differentiable~~ functions

Define: $X_0 : [0,\delta] \to \mathbb{R}$, $X_0(t) \equiv x_0$, $\forall t$.

Define a norm $\|x\| \triangleq \sup_{t\in[0,\delta]} |x(t)|$

Define a closed set S as an r-ball around the function x_0

$$S \triangleq \{ x(\cdot) \in C^n[0,\delta] \mid |x(\cdot) - x_0(\cdot)| \le r \}$$

Define a mapping $T: C^n[0,\delta] \to C^n[0,\delta]$

$$[T(x)](t) \triangleq (Tx)(t) \triangleq x_0 + \int_0^t f(x(\tau),\tau)\,d\tau$$

Goal 1: show that if δ is small enough, T is a contraction mapping on S.

Take any $x, y \in S$:

$$(Tx)(t) - (Ty)(t) = \int_0^t f(x(\tau),\tau) - f(y(\tau),\tau)\,d\tau$$

$$|(Tx)(t) - (Ty)(t)| \le \int_0^t K|x(\tau) - y(\tau)|\,d\tau$$

$$\|Tx - Ty\| \le \int_0^t K\|x-y\|\,d\tau \le K\delta\|x-y\|$$ so choose

$\delta < \frac{1}{K}$.

Goal 2: show that T maps S into S.

Take any $x \in S$: $$Tx(t) - x_0 = \int_0^t f(x(\tau),\tau)\,d\tau$$

$$= \int_0^t f(x(\tau),\tau) - f(x_0,\tau) + f(x_0,\tau)\,d\tau$$

$$|Tx(t) - x_0| \le \int_0^t K|x(\tau) - x_0|\,d\tau + \int_0^t |f(x_0,\tau)|\,d\tau$$

$$\le \underbrace{K\cdot r\cdot\delta + h\cdot\delta}_{\text{needs to be } \le r}$$

$K\cdot r\cdot\delta + h\delta \le r$, $\delta \le \frac{r}{Kr+h}$

thus pick $\delta < \min\left[\frac{1}{K}, \frac{r}{Kr+h}\right]$. Therefore (*) has a unique solution in S