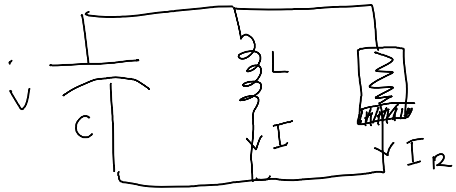
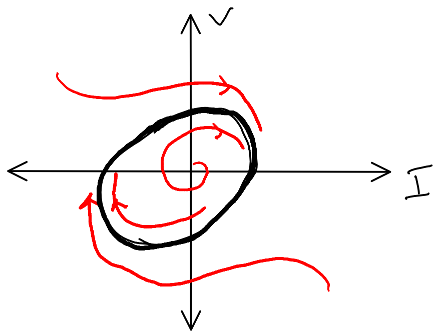
Limit cycles & Periodic Orbits

Consider the oscillator system from previous lectures:



$$\dot{I} = V$$
$$\dot{V} = -I - I_R(V).$$
$$= -I - V^3 + V$$

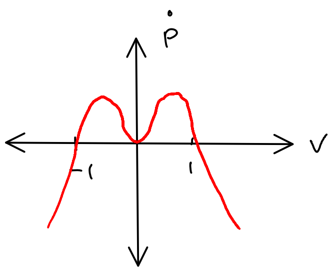


Specialties:
* robust oscillation
* steady state orbit does not depend on initial conditions

Potential Energy analysis:
Recall that the potential energy is given by:

$$P(I,V) = \frac{1}{2} I^2 + \frac{1}{2} V^2$$

Therefore: $$\frac{dP}{dt} = I \cdot \dot{I} + V \cdot \dot{V} = IV + V(-I - V^3 + V)$$
$$= -V^4 + V^2 = -V^2 (V-1)(V+1)$$



For small v ($\|v\| < 1$), potential energy increases. For large v, it decreases

Autonomous system $\dot{x} = F(x)$, $x \in \mathbb{R}^2$

Periodic motion: There exists a $T > 0$ such that

$$x(t+T) = x(t), \text{ for all } t$$

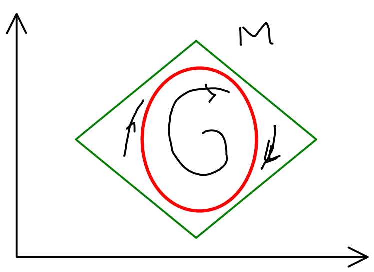
Two sufficient criteria about the existence of periodic orbits for planar systems:

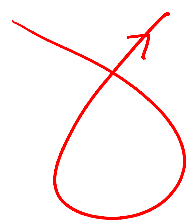
* Poincaré - Bendixson (existence)
* Bendixson (non existence)

Poincaré - Bendixson criterion: Suppose that a closed and bounded set $M \in \mathbb{R}^2$ is such that:

1. M does not contain any equilibrium or just one equilibrium which is an unstable node or focus, and
2. Every trajectory starting in M remains in M indefinitely,

then: M contains a periodic orbit



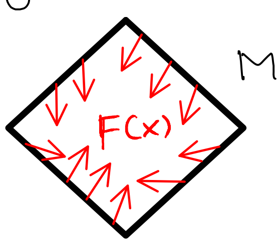


impossible!
a trajectory cannot cross itself. Why?

Example: Harmonic oscillator

Condition #2: M is invariant. How to check this?

A. Show that the vector field F(x) on the boundary of M points inward or tangent to the boundary.



B. Use a potential-like function $V(x)$. If $V(x)$ satisfies:

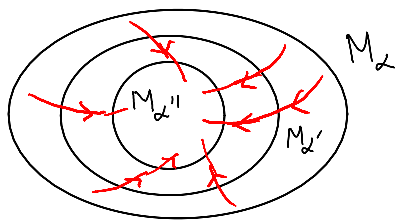
$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial V}{\partial x} \cdot F(x) \leq 0, \ \forall x$$

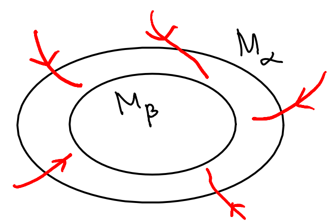
then the level sets $M_\alpha = \{ x \mid V(x) \leq \alpha \}$ is invariant for any α.

A weaker version:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial V}{\partial x} \cdot F(x) \leq 0, \ \forall x \text{ s.t } V(x) \geq \beta$$

then the level sets $M_\alpha = \{ x \mid V(x) \leq \alpha \}$ is invariant for $\alpha \geq \beta$





Example: [2.8 from the book]

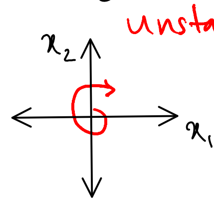
$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

There is only one equilibrium: $(0,0)$. The Jacobian is:

$$\frac{\partial F}{\partial x} = \begin{bmatrix} 1-(x_1^2+x_2^2) - 2x_1^2 & 1-2x_1x_2 \\ -2-2x_1x_2 & 1-(x_1^2+x_2^2)-2x_2^2 \end{bmatrix}$$

$$\left.\frac{\partial F}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \rightarrow \text{Eigenvalues} = 1 \pm j\sqrt{2}$$

unstable focus!



Consider the "potential" function : $V(x) = x_1^2 + x_2^2$.

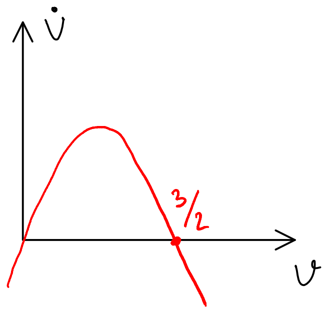
$$\frac{dV}{dt} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}^T \cdot F(x) = 2x_1^2 + 2x_1x_2 - 2x_1^2(x_1^2+x_2^2) + -4x_1x_2 + 2x_2^2 - 2x_2^2(x_1^2+x_2^2)$$

$$= 2x_1^2 + 2x_2^2 - 2(x_1^2+x_2^2)^2 - 2x_1x_2$$

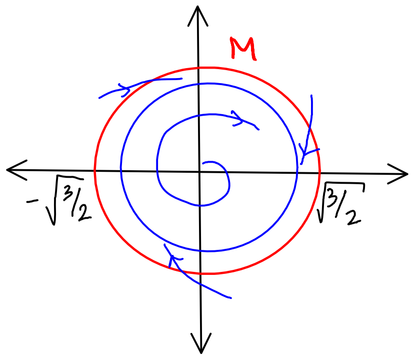
use $(-2x_1x_2 \leq x_1^2+x_2^2) \rightarrow$

$$\leq 3x_1^2 + 3x_2^2 - 2(x_1^2+x_2^2)^2$$

$$= 3V - 2V^2$$



Thus all level sets $V(x) \leq \alpha$ are invariant for $\alpha \geq 3/2$.



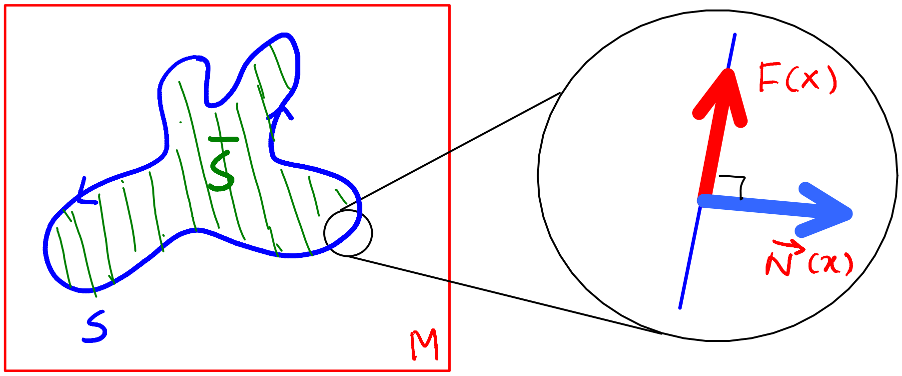
Using Poincaré - Bendixson criterion we can show that any M_α with $\alpha \geq 3/2$ contains a periodic orbit.

Bendixson criterion : Take a planar system

$$\dot{x}_1 = F_1(x_1, x_2)$$
$$\dot{x}_2 = F_2(x_1, x_2)$$

If $\left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right)$ is not identically zero and does not change sign in a simply connected region M, then M does not contain any periodic orbit.

Proof: By contradiction. Suppose that M contains a periodic orbit S.



$$\int_S \vec{F}(x)\cdot\vec{N}(x)\,dx = 0 \quad \text{(Integrate the flux of } F(x) \text{ along } S\text{)}$$

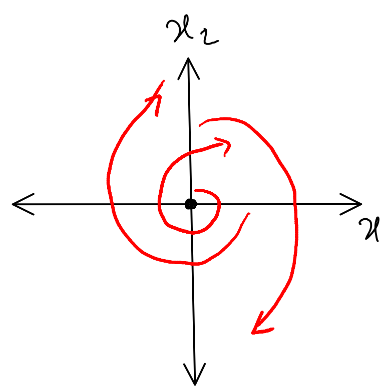
But, according to Green's theorem:

$$\int_S \vec{F}(x)\cdot\vec{N}(x)\,dx = \iint_{\bar{S}} \operatorname{div}(F)\,dx\,dy$$

$$= \iint_{\bar{S}} \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}\right) dx\,dy \neq 0$$

Examples: $\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - x_1^3 + x_2$



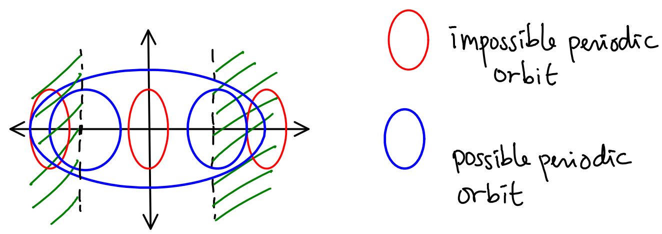
$\operatorname{div}(F) = 1 > 0 \rightarrow$ no periodic orbit

Example: $\dot{x}_1 = x_2$
$\dot{x}_2 = -\delta x_2 + x_1 - x_1^3 + x_1^2 x_2$

Equilibria: $(-1,0)$, $(0,0)$, $(1,0)$

$\mathrm{div}(F) = -\delta + x_1^2$ $\longrightarrow$ If $\delta < 0$, no periodic orbit

If $\delta \geq 0$



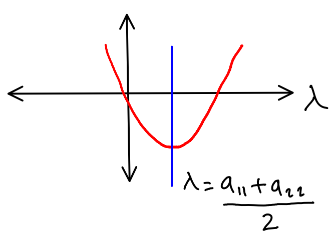
Application to linear systems: Let $\dot{x} = Ax$, where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\mathrm{div}(F) = a_{11} + a_{22}$, thus periodic orbits are possible only if

$$\boxed{a_{11} + a_{22} = 0}$$

Graphical explanation: The characteristic polynomial $\det(\lambda I - A)$
$= (\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21}$

$= \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$

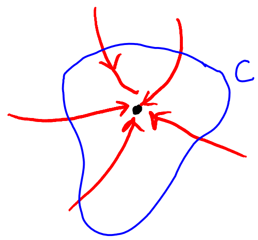


Index Theory

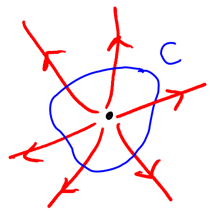
$$\dot{x} = F(x), \quad x \in \mathbb{R}^2$$

Take any closed curve C in $\mathbb{R}^2$ that does not pass through any equilibrium (not necessarily a trajectory). The index of C is the number of full circles made by $F(x)$ (in counterclockwise direction) as we traverse C counter clockwise.

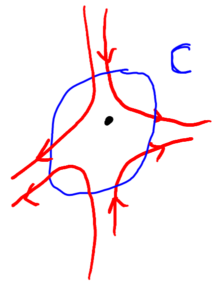
Examples:



Index $C = 1$



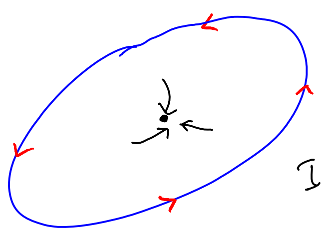
Index $C = 1$



Index $C = -1$

Lemma: The index of a node, focus, or a center is $+1$
The index of a saddle point is -1
The index of a closed trajectory is $+1$
The index of a closed curve is the sum of the indices of the equilibria in it.

Corollary: Inside any closed trajectory, there must be a non-saddle point equilibrium.



Index $= +1$

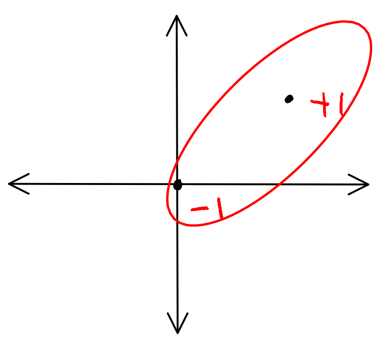
Example: $\dot{x}_1 = -x_1 + x_1 x_2$
$\dot{x}_2 = x_1 + x_2 - 2x_1 x_2$

There are two equilibria $(0,0)$ and $(1,1)$

Computing the Jacobian: $\frac{\partial F}{\partial x} = \begin{bmatrix} -1+x_2 & x_1 \\ 1-2x_2 & 1-2x_1 \end{bmatrix}$

$$\left.\frac{\partial F}{\partial x}\right|_{x=(0,0)} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \rightarrow \text{eigenvalues} = \pm 1 \quad \text{saddle point}$$

$$\left.\frac{\partial F}{\partial x}\right|_{x=(1,1)} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \rightarrow \text{eigenvalues} = \frac{-1 \pm j\sqrt{3}}{2} \quad \text{stable}$$



There is no periodic orbits that encircle both equilibria