Planar Dynamical Systems

Wednesday, January 21, 2009
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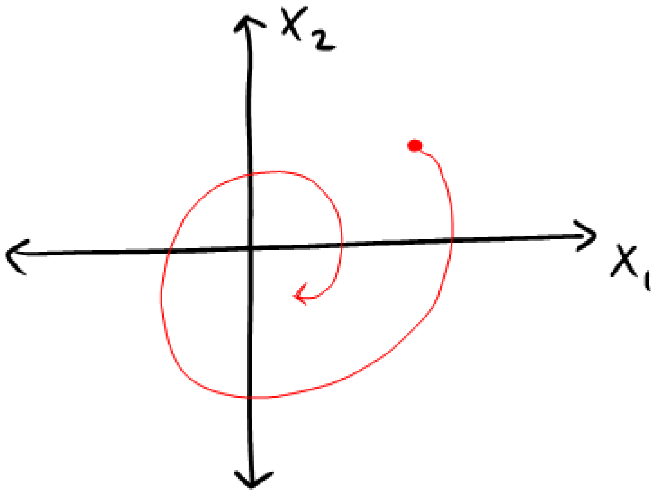
Consider a special case of autonomous dynamical systems where the state space is 2D.

$$\dot{x}_1 = f_1(x_1, x_2)$$

$$\dot{x}_2 = f_2(x_1, x_2)$$



$$\dot{x} = F(x), \; x \in \mathbb{R}^2$$

Example: van der Pol oscillator
Predator - prey model
Toggle switch

} from previous lecture

Special case: Linear Systems

$$\dot{x} = Ax \;, \quad A \in \mathbb{R}^{2\times 2}$$

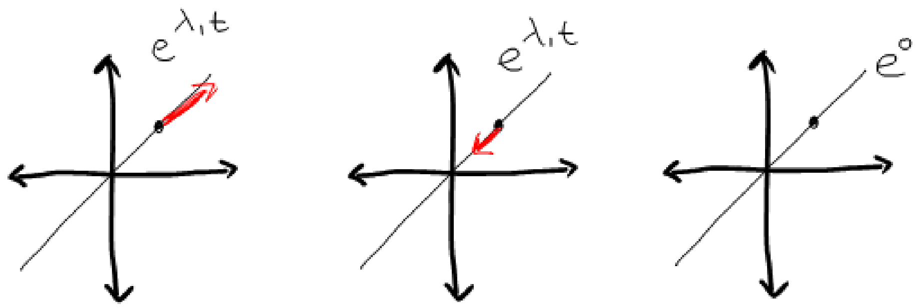
Depending on the eigenvalues and eigenvectors, there are different qualitative behaviors of the systems.

Case I: Both eigenvalues λ_1, λ_2 are real and distinct.

If $\lambda_1 > 0$, then the eigenspace is unstable (a)

$\lambda_1 < 0$, then the eigenspace is stable (b)

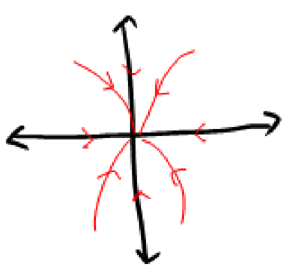
$\lambda_1 = 0$, then the eigenspace is marginally stable (c)



(a) (b) (c)

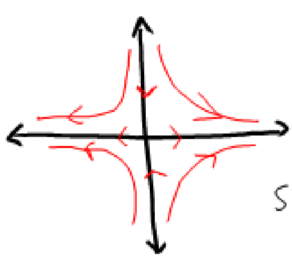
Ramifications:

(a) Two stable eigenvalues



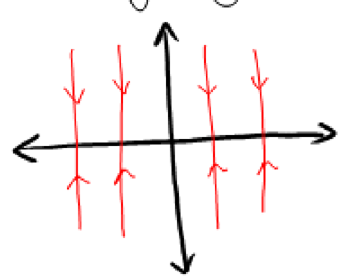
Two unstable eigenvalues: reverse the arrow

(b) One unstable eigen value



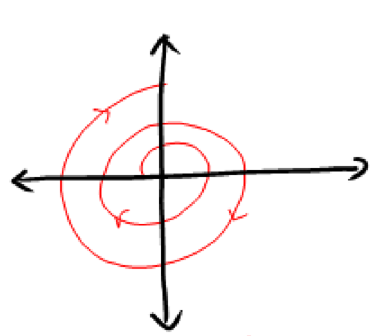
saddle point

(c) one marginally stable eigenvalue

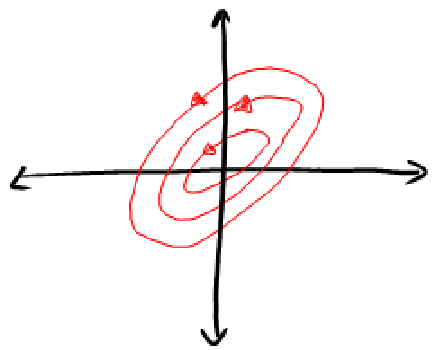


Case II: Complex eigenvalues $\lambda_{1,2} = a \pm jb$

If $a > 0 \rightarrow$ spiralling out / unstable oscillation
$a < 0 \rightarrow$ spiralling in / stable oscillation
$a = 0 \rightarrow$ stable sustained oscillation



unstable oscillation
$a > 0$



$a = 0$

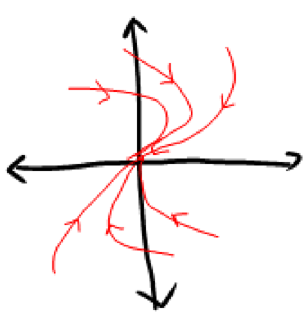
How do you tell if the oscillation is clockwise?

Case III: $\lambda_1 = \lambda_2$, but with distinct eigenspaces
see Case I

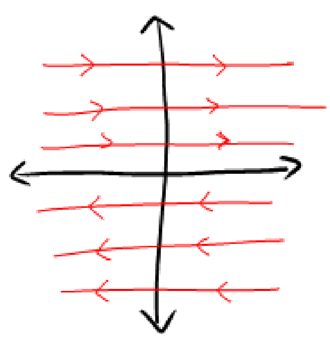
Case IV: $\lambda_1 = \lambda_2$ with shared eigenspace

$$A = \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}$$

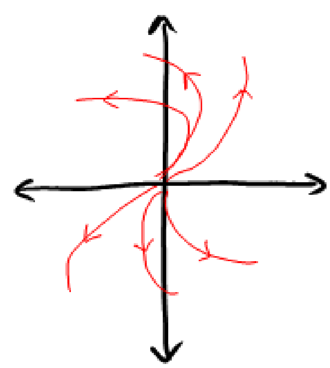
$\lambda \geq 0$ unstable
$\lambda < 0$ stable



$\lambda < 0$



$\lambda = 0$



$\lambda > 0$

Nonlinear Systems

Consider a more general planar nonlinear system

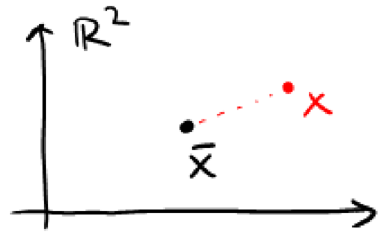
$$\dot{x} = F(x), \quad x \in \mathbb{R}^2$$

Let $\bar{x}$ be an equilibrium, i.e. $F(\bar{x}) = 0$

$$\bar{x} = (\bar{x}_1, \bar{x}_2)$$

Linearization

$$\dot{x}_1 = F_1(x_1, x_2)$$
$$\dot{x}_2 = F_2(x_1, x_2)$$



$$F_1(x_1, x_2) = F_1(\bar{x}_1, \bar{x}_2) + \left.\frac{\partial F_1}{\partial x_1}\right|_{x=\bar{x}} \cdot (x_1 - \bar{x}_1) + \dots + \left.\frac{\partial F_1}{\partial x_2}\right|_{x=\bar{x}} \cdot (x_2 - \bar{x}_2) + h.o.t$$

$$= \left.\frac{\partial F_1}{\partial x_1}\right|_{x=\bar{x}} \cdot (x_1 - \bar{x}_1) + \left.\frac{\partial F_1}{\partial x_2}\right|_{x=\bar{x}} \cdot (x_2 - \bar{x}_2) + h.o.t$$

Similarly:

$$F_2(x_1, x_2) = \left.\frac{\partial F_2}{\partial x_1}\right|_{x=\bar{x}} \cdot (x_1 - \bar{x}_1) + \left.\frac{\partial F_2}{\partial x_2}\right|_{x=\bar{x}} \cdot (x_2 - \bar{x}_2) + h.o.t$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix}_{x=\bar{x}}}_{\text{Jacobian matrix}} \cdot \begin{pmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{pmatrix} + h.o.t$$

Nonlinear systems can be approximated by linear systems

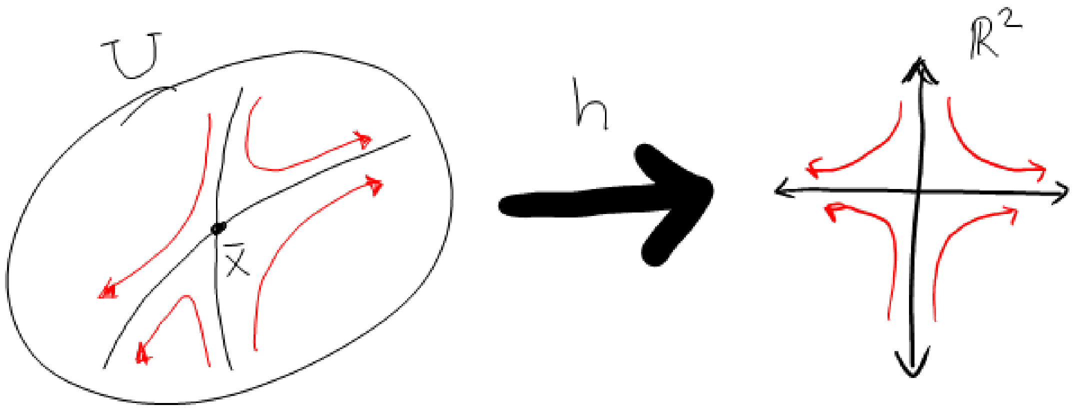
Hartman - Grobman Theorem

For a planar dynamical system with equilibrium $\bar{x}$, if the eigenvalues of the Jacobian at $\bar{x}$ exclude the imaginary axis, then there is a neighborhood U around $\bar{x}$ and a homeomorphism

$$h : U \to \mathbb{R}^2$$

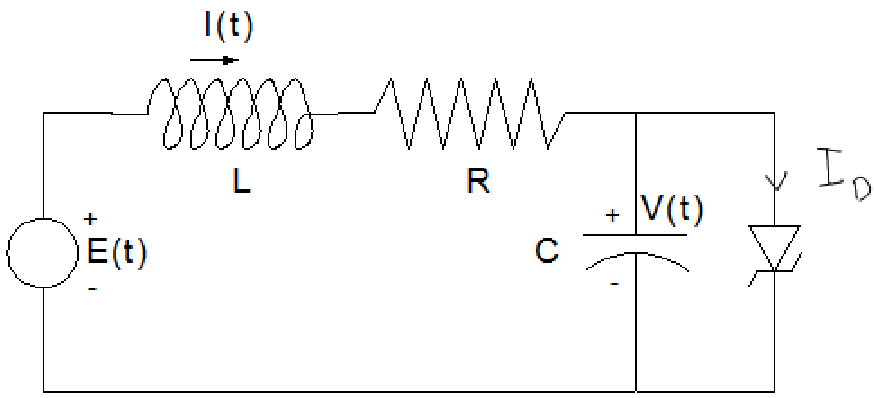
that maps the nonlinear trajectories in U to the linear trajectories in $\mathbb{R}^2$

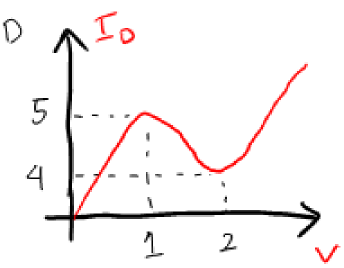
Note: Homeomorphism = continuous and invertible mapping



Examples from previous lecture

1) Tunnel diode circuit





$$\dot{I} = (E - RI - V)\cdot\frac{1}{L}$$

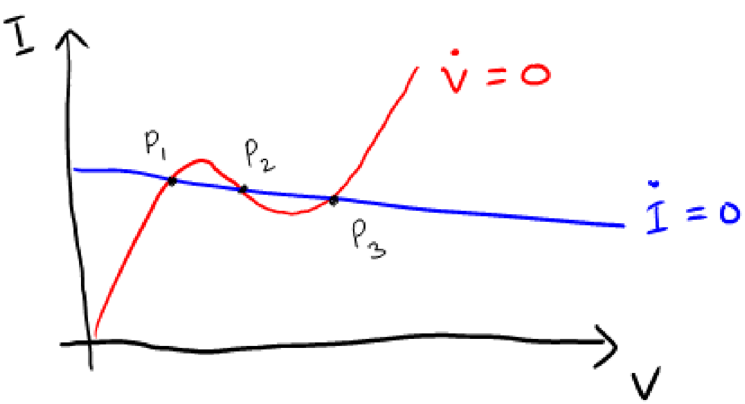
$$\dot{V} = (I - I_D(V))\frac{1}{C}$$

$L = C = 1$
$R = 10$
$E = 4.5$

$$I_D(V) = 2V^3 - 9V^2 + 12V$$

At equilibria: $I = I_D(V) = 2V^3 - 9V^2 + 12V$

$$4.5 - 10I - V = 0$$



Three equilibria

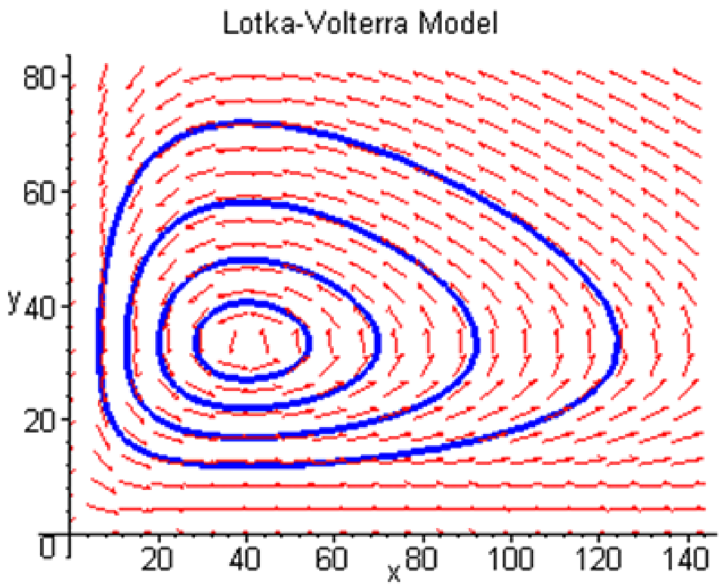
P_1, P_2, and P_3

$$4.5 - 20V^3 + 90V^2 - 121V = 0$$

The Jacobian : $\frac{\partial F}{\partial x} = \begin{bmatrix} -10 & -1 \\ 1 & -6V^2+18V-12 \end{bmatrix}$

Compute the eigenvalues for different equilibria and see that P_1 and P_3 are stable nodes, while P_2 is unstable.

2) Predator - prey model



$\dot{x} = ax - bxy$

$\dot{y} = cxy - dy$

Equilibrium:

$x = \frac{d}{c}$; $y = \frac{a}{b}$

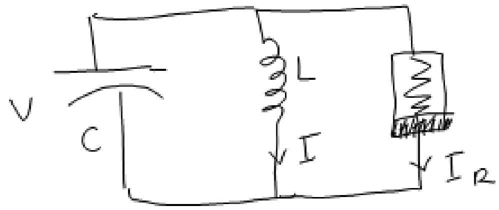
$x = 0$; $y = 0$

Jacobian $= \begin{pmatrix} a-by & -bx \\ cy & cx-d \end{pmatrix} \equiv \frac{\partial F}{\partial \xi}$ where $\xi = \begin{pmatrix} x \\ y \end{pmatrix}$

At $x=0, y=0$: $\frac{\partial F}{\partial \xi} = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix}$ saddle point

At $x = \frac{d}{c}, y = \frac{a}{b}$: $\frac{\partial F}{\partial \xi} = \begin{pmatrix} 0 & -\frac{bd}{c} \\ \frac{ca}{b} & 0 \end{pmatrix}$ Inconclusive

3) Van der Pol oscillator



$$L\dot{I} = V$$
$$C\dot{V} = -I - I_R(V).$$

$L = C = 1$

$I_R(V) = V^3 - V$

Only one equilibrium at $(0,0)$

The Jacobian $= \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \rightarrow$ eigenvalues $= \frac{1}{2} \pm j\,\frac{1}{2}\sqrt{3}$

The local dynamics is unstable oscillation.

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