Chapter 13 FEEDBACK LINEARIZATION.

Today: input-output linearization

SISO nonlinear system: $\begin{cases} \dot{x} = f(x) + g(x) u \\ y = h(x) \end{cases}$

Relative degree (r):

Linear: $\dfrac{s^m + a_{m-1} s^{m-1} + \dots a_0}{s^n + b_{n-1} s^{n-1} + \dots + b_0}$ $r = n - m$.

The rel. deg. (r): # of times we need to differentiate the output to see the input (u)

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x) u]$$

$$= \frac{\partial h}{\partial x} f(x) + \underbrace{\frac{\partial h}{\partial x} g(x)}_{\neq 0} u \qquad r = 1$$

If $\frac{\partial h}{\partial x} g(x) = 0$,

$$\ddot{y} = \frac{d}{dt}\left(\frac{\partial h}{\partial x} f(x)\right)$$

Lie derivative: $L_f h(x) := \frac{\partial h}{\partial x} f(x)$

$$\ddot{y} = L_f L_f h(x) + \underbrace{L_g L_f h(x)}_{\neq 0 \quad r=2} \cdot \frac{d}{dt}\left[\frac{\partial h}{\partial x} f(x)\right]$$

$\doteq 0$, continue differentiating

Examples.

- $\dot{x}_1 = x_2 \quad \dot{x}_2 = -x_1^3 + u \quad y = x_1$

  $\dot{y} = \dot{x}_1 = x_2 \quad \ddot{y} = \dot{x}_2 = -x_1^3 + u \quad r = 2.$

- Linear system: $\dot{x} = Ax + Bu \quad y = Cx$

  $\dot{y} = C\dot{x} = CAx + CBu$ if $CB \neq 0$, $r = 1$

  $CB = 0 \Rightarrow \ddot{y} = CA\dot{x} = CA(Ax+Bu) = CA^2x + CABu$

  $CAB \neq 0 \Rightarrow r = 2$. $CAB = 0 \Rightarrow \dddot{y}$

  $CB, CAB, CA^2B, \cdots, CA^{r-1}B$

  rel. deg. $= r \Leftrightarrow CB = CAB = \cdots = CA^{r-2}B = 0$

  $CA^{r-1}B \neq 0$

- $\dot{x}_1 = x_2 + x_3^3$, $\dot{x}_2 = x_3$, $\dot{x}_3 = u$

  $y = x_1$

$\dot{y} = \dot{x}_1 = x_2 + x_3^3 \qquad \ddot{y} = x_3 + 3x_3 u$

$x_3 = 0$

The rel. deg. is NOT well defined at the origin.

Def'n $\dot{x} = f(x) + g(x)u$, $y = h(x)$, has relative degree r in a region D if

$$\begin{cases} L_g L_f^{i-1} h(x) = 0, & i = 1, \cdots, r-1 \\ L_g L_f^{r-1} h(x) \neq 0 \end{cases}$$

satisfied for all x in D.

Def'n. If a system has a well-defined relative degree, then it is "I-O Linearizable" because

$$y^{(r)} = L_f^{(r)} h(x) + \underbrace{L_g L_f^{(r-1)} h(x)}_{\neq 0}\, u$$

$$u = -\frac{1}{L_g L_f^{(r-1)} h(x)} \cdot L_f^{(r)} h(x) + v$$

$$y^{(r)} = v \qquad v \to \underbrace{\boxed{\int} \to \boxed{\int} \to \cdots \boxed{\int}}_{r \text{ integrators}} \to y$$

$$u = -\frac{1}{L_g L_f^{(r-1)} h(x)} \cdot L_f^{(r)} h(x) + v(t)$$

<u>zero dynamics</u>:

zero your output $y \equiv 0$, $v(t) = 0$

$$\Downarrow$$

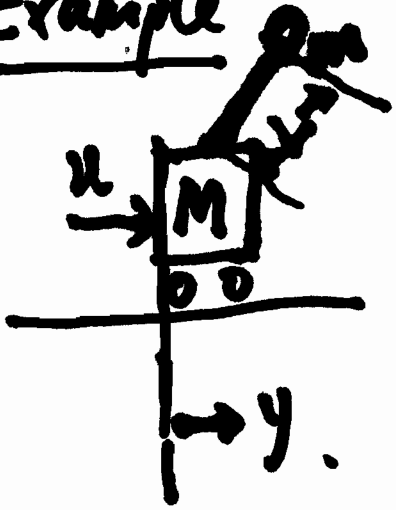
$y(0) = 0$, $y'(0) = 0$, $\cdots$, $y^{(r-1)}(0) = 0$, $y^{(r)} = 0$

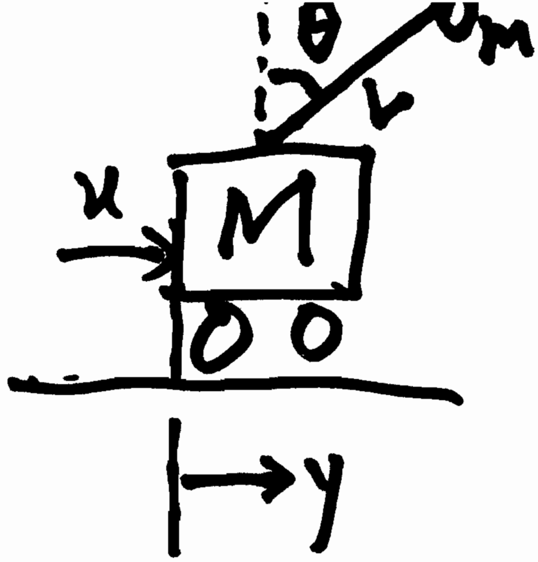
When $y \equiv 0$, the remaining $(n-r)$ order dynamics on the surface $\underline{y = \dot{y} = \ddot{y} = \cdots y^{(r-1)} = 0}$ are called zero dynamics.

If the origin of the zero dynamics is a.s., the system is "minimum phase".

If not, "non-minimum phase" system

Example





$$\ddot{y} = \frac{1}{\frac{M}{m}+\sin^2\theta}\left[\frac{u}{m}+\dot{\theta}^2 L\sin\theta - g\sin\theta\cos\theta\right]$$

$$\ddot{\theta} = \frac{1}{L\left(\frac{M}{m}+\sin^2\theta\right)}\left[-\frac{u}{m}\cos\theta-\dot{\theta}^2 L\cos\theta\sin\theta + \frac{M+m}{m} g\sin\theta\right]$$

$(y, \theta, \dot{y}, \dot{\theta})$ 4 states, 4^{th} order

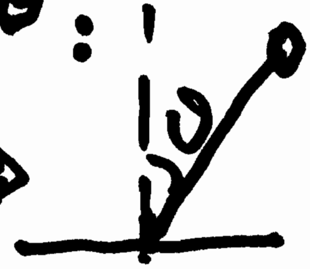
$r = 2$ zero dynamics: 2^{nd} order.
$(4-2)$

$y = \dot{y} = \ddot{y} = 0.$

$\ddot{y} = 0 \Leftrightarrow u^* = -m(\dot{\theta}^2 L\sin\theta - g\sin\theta\cos\theta)$

Substitute u^* to $\ddot{\theta}$:

$$\ddot{\theta} = \frac{g}{L}\sin\theta \Rightarrow$$



$\theta = 0$ is not stable, we have "non-minimum phase".

CANONICAL FORMS

Linear system: $\exists\ T$ s.t. $Tx = \begin{bmatrix} \eta \\ \xi \end{bmatrix}$, $\eta \in \mathbb{R}^{n-r}$, $\xi \in \mathbb{R}^{r}$

s.t. $\dot{\eta} = A_0\eta + B_0\xi$,

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \xi_3$$

$$\vdots$$

$$\dot{\xi}_r = k^T\xi + C_0\eta + \beta u, \quad \beta \neq 0$$

$$y = \xi_1$$

$y \equiv 0$, zero dynamics: $\dot{\eta} = A_0\eta$

Eigenvalues of A_0 are zeros of the transfer function.

Nonlinear system: If rel. deg. $r \le n$,

then you can find change of variables

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x).$$

$$\begin{cases} \dot{\eta} = f_0(\eta, \xi) \\ \dot{\xi}_1 = \xi_2 \\ \vdots \\ \dot{\xi}_r = \alpha(\eta, \xi) + \underbrace{\gamma(\eta, \xi)}_{\neq 0} u \end{cases}$$

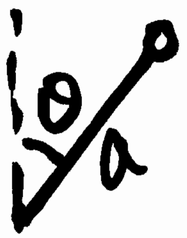
$y = \xi_1$

zero $y \equiv 0$, zero dynamics: $\dot{\eta} = f_0(\eta)$

$\boxed{\text{THM 13.1}}$

Full-State Linearization

Example:



$\theta = x_1$

$\dot\theta = x_2$

$$\dot x_1 = x_2$$
$$\dot x_2 = a\sin x_1 - b x_2 + c u$$
$$a, b, c > 0$$

Design u s.t. $x_1 = 0$, $x_2 = 0$ is a.s.

$$u = \frac{1}{c}[-a\sin x_1] + \frac{1}{c} v$$
$$\Downarrow$$
$$\dot x_1 = x_2$$
$$\dot x_2 = -b x_2 + v$$

$$v = -k_1 x_1 - k_2 x_2$$

$$\dot x_1 = x_2$$
$$\dot x_2 = -k_1 x_1 - (k_2 + b) x_2, \quad k_1, k_2 > 0$$

More generally.

$$\dot x = Ax + B[\alpha(x) + \gamma(x) u],$$

where $\gamma(x)$ is nonsingular

then, the preliminary feedback

$$u = \frac{1}{\gamma(x)}\left[-\alpha(x) + v\right] \text{ linearizes}$$

the system: $\dot{x} = Ax + Bv$

What if u is not matched to $\alpha(x)$?

Example:
$$\dot{x}_1 = x_2 + x_2^3$$
$$\dot{x}_2 = -x_1^2 + u$$

Changing variables: $x_1 = z_1$, $z_2 = \dot{x}_1 = x_2 + x_2^3$

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \dot{x}_2 + 3x_2^2\dot{x}_2 = \underbrace{-(1+3x_2^2)x_1^2}_{\alpha(x)} + \underbrace{(1+3x_2^2)}_{\gamma(x)}u$$

$$u = \frac{1}{\gamma(x)}\left[-\alpha(x) + v\right]$$

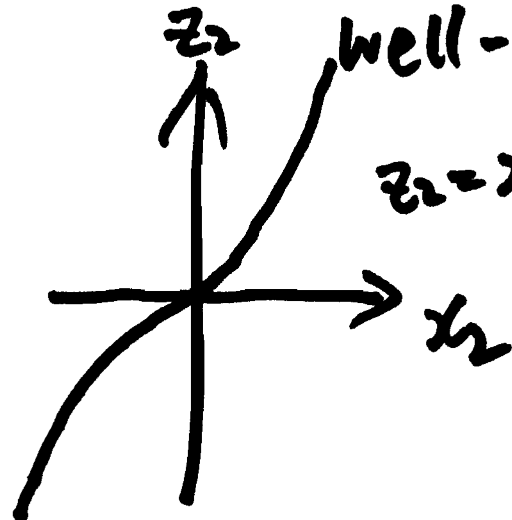
$$\Rightarrow \begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = v \end{cases}$$

- Is this change of variables valid?

- $z = T(x)$ T^{-1} exists
- $T(\cdot)$, $T^{-1}(\cdot)$ is C^1 [($\dot{z}$) equation should be well-defined]

$z_1 = x_1$

$z_2 = x_2 + x_2^3$



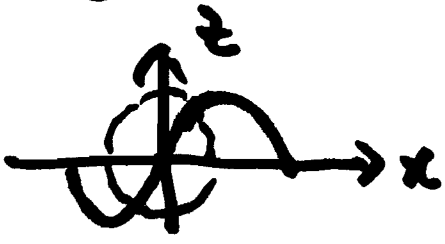
$z_2 = x_2 + x_2^3$

$\frac{\partial z_2}{\partial x} = 1 + 3x_2^2$

In this example. T is valid!

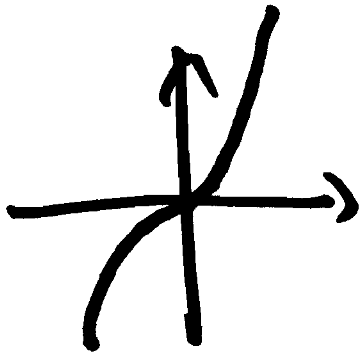
<u>Examples</u>:

- $z = Tx$, T is nonsingular matrix.
- $z = \sin x$



local.

- $z = x^3$



$x = z^{\frac{1}{3}}$ not C^1

X

- $z = x + x^3$ valid.
- Such a valid transformation is called a diffeomorphism!

<u>Lemma</u> T is a local diffeomorphism in a neighborhood of x_0 iff $\frac{\partial T}{\partial x}$ is not singular at x_0.

It is a global diffeo. if, in addition, $\lim_{\|x\| \to \infty} \|T\| = \infty$.

Proof: Implicit function theorem!

<u>Defn</u>: A nonlinear system $\dot{x} = f(x) + g(x)u$ is feedback linearizable if $\exists$ a diffeomorphism $z = T(x)$, s.t.

$$\dot{z} = Az + B[\alpha(x) + \gamma(x)u]$$

where $\gamma(x)$ is nonsingular, (A, B) controllable