Integral Control

When the parameters are not exactly known, proportional feedback can result in steady state error.

Consider the plant:

$$\dot{x} = f(x,u,w) \quad , x \in \mathbb{R}^n \quad , u \in \mathbb{R}^p$$
$$y = h(x,w) \quad , y \in \mathbb{R}^p$$
$$y_m = h_m(x,w) \quad , y_m \in \mathbb{R}^m$$

y_m is the measured outputs, and y is a subset of y_m. w is a set of unknown parameters.

We want to design a feedback control such that $\lim_{t\to\infty} y(t) = r$, for some $r \in \mathbb{R}^p$

Assume that for each parameter w and reference r, there exist u_{ss} and x_{ss} such that uniquely:

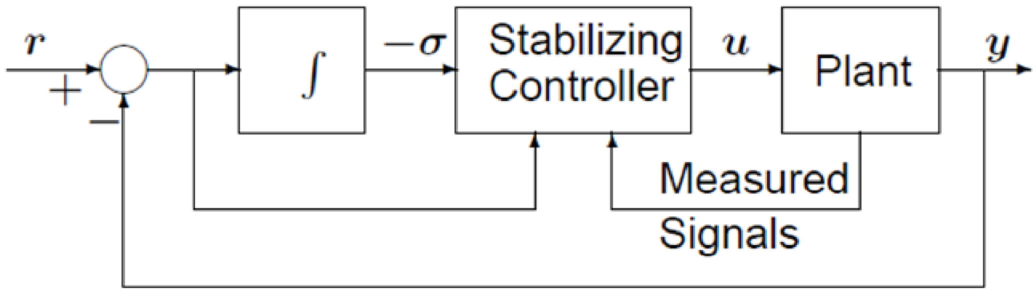
$$f(x_{ss}, u_{ss}, w) = 0$$
$$h(x_{ss}, w) = r$$

Idea: introduce integral action to drive the tracking error

$$e \triangleq y - r$$

asymp. to zero, i.e. $\dot{\sigma} = e$. Thus the controller to be designed is:

$$u = \gamma(y_m, e, \sigma)$$



Full State Feedback ($y_m = x$)

The controller: $u = -K_1 x - K_2 \sigma - K_3 e$

The closed loop system:

$$\dot{x} = f(x, -K_1 x - K_2 \sigma - K_3(h(x,w) - r), w)$$
$$\dot{\sigma} = h(x,w) - r$$

At equilibrium: (x_{eq}, σ_{eq}) we have:

$$\boxed{\begin{aligned} f(x_{eq}, -K_1 x_{eq} - K_2 \sigma_{eq}, w) &= 0 \\ h(x_{eq}, w) - r &= 0 \end{aligned}}$$

By assumption: $x_{eq} = x_{ss}$

$$U_{eq} = -K_1 x_{eq} - K_2 \sigma_{eq} = U_{ss}$$

σ_{eq} is unique if K_2 is nonsingular

$\boxed{\text{Task: Stabilize the system around } (x_{eq}, \sigma_{eq})}$

Define
$$\tilde{x} \triangleq x - x_{eq}$$
$$\tilde{\sigma} \triangleq \sigma - \sigma_{eq}$$

Linearization:

$$\left.\begin{aligned} \dot{\tilde{x}} &= A_w \tilde{x} + B_w(-K_1 \tilde{x} - K_2 \tilde{\sigma}), \text{ where} \\ \dot{\tilde{\sigma}} &= C_w \tilde{x} \end{aligned}\right\} \quad (*)$$

$$A_w \triangleq \frac{\partial}{\partial x} f(x,u,w)\Big|_{x = x_{ss};\, u = u_{ss}} \quad ; \quad B_w \triangleq \frac{\partial}{\partial u} f(x,u,w)\Big|_{x = x_{ss},\, u = u_{ss}}$$

$$C_w \triangleq \frac{\partial}{\partial x} h(x,w)\Big|_{x = x_{ss}}$$

We can rewrite $(*)$ as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\sigma}} \end{pmatrix} = \left[\underbrace{\begin{pmatrix} A_w & 0 \\ C_w & 0 \end{pmatrix}}_{\mathcal{A}} - \underbrace{\begin{pmatrix} B_w \\ 0 \end{pmatrix}}_{\mathcal{B}} \underbrace{\begin{pmatrix} K_1 & K_2 \end{pmatrix}}_{K} \right] \begin{pmatrix} \tilde{x} \\ \tilde{\sigma} \end{pmatrix}$$

A stabilizing K exists iff $(\Lambda, \mathcal{B})$ is stabilizable.

1) (A_w, B_w) is stabilizable
2) $\begin{pmatrix} A_w & B_w \\ C_w & 0 \end{pmatrix}$ is fullrank

Notice that stabilization needs to be done independent of w.

Example: The pendulum model: $\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT$

Suppose that we do not know the values of a, b, c, except that: a, b, c > 0 and

$$\frac{a}{c} < \rho_1 \quad \text{and} \quad \frac{1}{c} < \rho_2$$

We want to stabilize θ at δ. Therefore, taking $x_1 \triangleq \theta - \delta$, $u \triangleq T$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin(x_1+\delta) - bx_2 + cu$$
$$y = x_1 \quad ; \lim_{t\to\infty} y(t) = 0$$
$$w = (a, b, c)^T$$

$x_{ss} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$; $u_{ss} = \frac{a}{c}\sin\delta$

$A_w = \begin{pmatrix} 0 & 1 \\ -a\cos\delta & -b \end{pmatrix}$; $B_w = \begin{pmatrix} 0 \\ c \end{pmatrix}$; $C_w = \begin{pmatrix} 1 & 0 \end{pmatrix}$

We know that (A_w, B_w) is controllable, and

$\begin{pmatrix} A_w & B_w \\ C_w & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -a\cos\delta & -b & c \\ 1 & 0 & 0 \end{pmatrix}$ is fullrank iff $c \neq 0$. Thus, for any value of w, we know that there exists a stabilizing K. However, we need to design K independent of w.

$\Lambda = \begin{pmatrix} 0 & 1 & 0 \\ -a\cos\delta & -b & 0 \\ 1 & 0 & 0 \end{pmatrix}$; $\mathcal{B} = \begin{pmatrix} 0 \\ c \\ 0 \end{pmatrix}$; $K = \begin{pmatrix} k_1 & k_2 & k_3 \end{pmatrix}$

$$A - BK = \begin{pmatrix} 0 & 1 & 0 \\ -a\cos\delta - ck_1 & -b - ck_2 & -ck_3 \\ 1 & 0 & 0 \end{pmatrix}.$$ Compute the characteristic polynomial and run the Routh-Hurwitz test.

Conditions for stability:

$b + k_2 c > 0 \quad \to k_2 > 0$

$k_3 c > 0 \quad \to k_3 > 0$

$(b + k_2 c)(a\cos\delta + k_1 c) - k_3 c > 0$

If $(k_1 c > a)$ then $(a\cos\delta + k_1 c) > 0$, thus pick $k_1 > \frac{a}{c}$, which is guaranteed if $k_1 > \rho_1$

Further, if $k_1 k_2 c^2 > k_3 c$ then stability is guaranteed, thus make sure that $k_1 > \frac{k_3}{k_2} \cdot \frac{1}{c}$ by choosing $k_1 > \frac{k_3}{k_2} \cdot \rho_2$

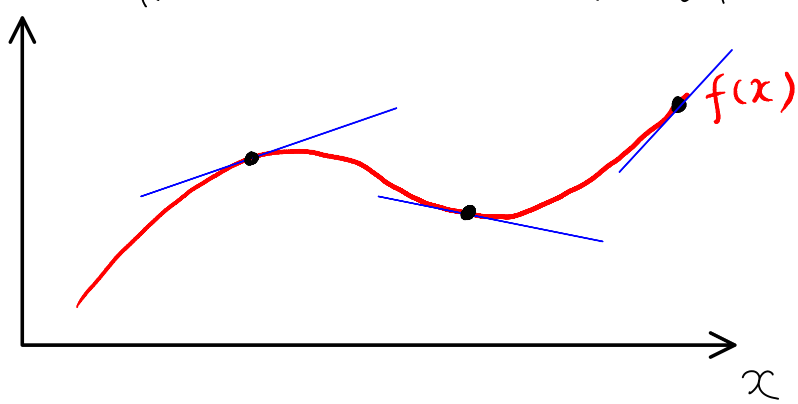
Therefore $k_1 > \max\left(\rho_1, \frac{k_3}{k_2}\rho_2\right)$

Gain scheduling: (see ch 12.5)

Linearization based design techniques are local by default. How to extend the locality? Eg: If the desired equilibrium is time varying

Idea: Parameterized the state space with scheduling variable, for each scheduling variable, design a linearization based controller based on the scheduling variable

Switch to different controllers as the operating point changes.



Note: Works well if the scheduling variable does not change too fast

Widely applied, e.g. in flight control