Nonlinear Feedback Control

Consider a plant system: $\dot{x} = f(x, u, t)$

Stabilization problem: Design a feedback control such that the system is stabilized at $x = x_{ss}$.

We can generally assume that $x_{ss} = 0$, through coordinate change.

Types of stabilization:

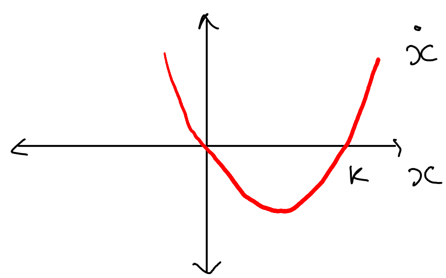
- Local: unknown region of attraction
- Regional: region of attraction is guaranteed to include a compact set G
- Semiglobal: feedback control can be designed such that the r.o.a includes any compact set.
- Global

Example: $\dot{x} = x^2 + u$

Suppose that $u = -Kx$, thus $\dot{x} = x^2 - Kx$, $K > 0$

Through linearization at the origin, we know that the origin is locally asymp. stable.

Plot $\dot{x}$ vs x:



After further analysis, we know that the region of attraction is $\{x < K\}$, thus we can achieve regional stabilization. Further, since K can be chosen arbitrarily, the linear feedback control achieves semiglobal stabilization.

A different feedback control scheme: $u = -x^2 - Kx$ results in

$$\dot{x} = -Kx$$

and thus achieves global stabilization.

Stabilization by linearization

Consider the plant : $\dot{x} = f(x,u)$, where $f(0,0) = 0$, f is cont. diff.
$x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$

Define : $A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0}$, $B \triangleq \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0}$

If we introduce a linear feedback : $u = -Kx$, the closed loop system becomes
$\dot{x} = f(x, -Kx) \triangleq \tilde{f}(x)$

Notice that $\tilde{f}(0) = 0$, and $\frac{\partial \tilde{f}}{\partial x} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial u} \cdot K$, thus

$$\left.\frac{\partial \tilde{f}}{\partial x}\right|_{x=0} = A - BK$$

Thus, if K is chosen such that $(A-BK)$ is Hurwitz, then we achieve local asymp. stability.

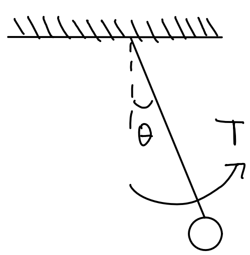
Lemma : Such K exists iff the pair (A,B) is stabilizable

Hautus test : (A,B) is stabilizable if the polynomial matrix $[sI - A \,\vdots\, B]$ has rank n for all $\mathrm{Re}\, s \geq 0$

Lemma : If (A,B) is controllable then a stabilizing feedback K exists.

Example : Pendulum system : $\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT$, $a,b,c > 0$

Suppose that we want to stabilize the pendulum at $\theta = \delta$



Define the necessary constant torque to maintain $\theta = \delta$ as T_{ss}. We have $T_{ss} = \frac{a}{c}\sin\delta$

Define : $x_1 = \theta - \delta$
$x_2 = \dot{x}_1$
$u = T - T_{ss}$

Then :
$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1 + \delta) - bx_2 + c\left(u + \tfrac{a}{c}\sin\delta\right) \end{aligned}\right\} = f(x,u)$$

$$A = \frac{\partial f}{\partial x}\Big|_{x=0,u=0} = \begin{pmatrix} 0 & 1 \\ -a\cos\delta & -b \end{pmatrix} ; B = \frac{\partial f}{\partial u}\Big|_{x=0,u=0} = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

Kalmar rank test: $[B \vdots AB] = \begin{bmatrix} 0 & c \\ c & -bc \end{bmatrix}$ is full rank if $c \neq 0$. Thus the linearization is controllable. Suppose that the linear feedback is:

$$u = -K_1 x_1 - K_2 x_2 \text{ , then}$$

$$A - BK = \begin{pmatrix} 0 & 1 \\ -a\cos\delta - cK_1 & -b - cK_2 \end{pmatrix} \to \text{char. polynomial: } \lambda(\lambda + b + cK_2) + a\cos\delta + cK_1$$

The closed loop system is Hurwitz if
$$b + cK_2 > 0$$
$$a\cos\delta + cK_1 > 0$$

Output feedback

Given the plant: $\dot{x} = f(x, u)$, $y = h(x)$: $h(0) = 0$

Suppose that we want to achieve stabilization through output feedback.

Define: $C \triangleq \frac{\partial h}{\partial x}\Big|_{x=0}$

Through linearization, we obtain (locally):
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

If (A, C) is detectable, we can design an observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + F(y - C\hat{x}) \text{ , such that}$$

$$\begin{aligned}
\dot{x} - \dot{\hat{x}} &= A(x - \hat{x}) - FC(x - \hat{x}) \\
&= (A - FC)(x - \hat{x}), \text{ with } (A - FC) \text{ Hurwitz} \\
\dot{e} &= (A - FC)e
\end{aligned}$$

The controller $u = -K\hat{x}$

Thus the closed loop system is:
$$\dot{x} = Ax - BK\hat{x} = (A - BK)x + BKe$$
$$\dot{e} = (A - FC)e$$

$$\begin{pmatrix} \dot{x} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - FC \end{pmatrix}$$. Notice that the controller is <u>dynamic</u>.

Integral Control

When the parameters are not exactly known, proportional feedback can result in steady state error.

Consider the plant:

$$\dot{x} = f(x, u, w) \quad , x \in \mathbb{R}^n \quad , u \in \mathbb{R}^p$$
$$y = h(x, w) \quad , y \in \mathbb{R}^p$$
$$y_m = h_m(x, w) \quad , y_m \in \mathbb{R}^m$$

y_m is the measured outputs, and y is a subset of y_m. w is a set of unknown parameters.

We want to design a feedback control such that $\lim_{t\to\infty} y(t) = r$, for some $r \in \mathbb{R}^p$

Assume that for each parameter w and reference r, there exist u_{ss} and x_{ss} such that uniquely:

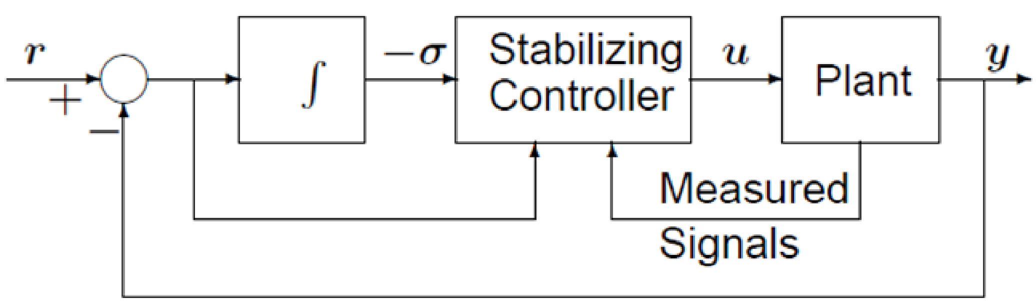
$$f(x_{ss}, u_{ss}, w) = 0$$
$$h(x_{ss}, w) = r$$

Idea: introduce integral action to drive the tracking error

$$e \triangleq y - r$$

asymp. to zero, i.e. $\dot{\sigma} = e$. Thus the controller to be designed is:

$$u = \gamma(y_m, e, \sigma)$$



Full State Feedback ($y_m = x$)

The controller: $u = -K_1 x - K_2 \sigma - K_3 e$

The closed loop system:

$$\dot{x} = f(x, -K_1 x - K_2 \sigma - K_3 (h(x,w) - r), w)$$
$$\dot{\sigma} = h(x,w) - r$$

At equilibrium: (x_{eq}, σ_{eq}) we have:

$$\boxed{\begin{aligned} f(x_{eq}, -K_1 x_{eq} - K_2 \sigma_{eq}, w) &= 0 \\ h(x_{eq}, w) - r &= 0 \end{aligned}}$$

By assumption: $x_{eq} = x_{ss}$

$$u_{eq} = -K_1 x_{eq} - K_2 \sigma_{eq} = u_{ss}$$

σ_{eq} is unique if K_2 is nonsingular

$$\boxed{\text{Task: Stabilize the system around } (x_{eq}, \sigma_{eq})}$$