Example: $\dot{x}_1 = -x_1 - 1.5x_1^2 x_2^3$
$\dot{x}_2 = -x_2^3 + 0.5x_1^2 x_2^2$

Decouple the system such that: $f_1(x_1) = -x_1$; $g_1(x) = 1.5x_1^2 x_2^3$
$f_2(x_2) = -x_2^3$; $g_2(x) = 0.5\, x_1^2 x_2^2$

For system 1, pick $V_1(x_1) = \frac{1}{2}x_1^2 \rightarrow \dot{V}_1 = x_1 \cdot -x_1 = -x_1^2$
$= -\alpha_1 \phi_1^2(x_1)$

where $\alpha_1 = 1$; $\phi_1(x_1) = |x_1|$, also $\left|\frac{\partial V_1}{\partial x_1}\right| \le \beta_1 \phi(x_1)$ with $\beta_1 = 1$

For system 2, pick $V_2(x_2) = \frac{1}{4}x_2^4 \rightarrow \dot{V}_2 = x_2^3 \cdot -x_2^3 = -x_2^6$
$= -\alpha_2 \phi_2(x_2)$

where $\alpha_2 = 1$, $\phi_2(x_2) = |x_2|^3$. Then $\left|\frac{\partial V_2}{\partial x_2}\right| = x_2^3 \le \beta_2 \phi_2(x_2)$, with $\beta_2 = 1$.

The coupling between subsystems can be bounded by:
$|g_1(x)| \le 1.5c_1^2 \phi_2(x_2)$ if $|x_1| \le c_1$

$|g_2(x)| \le 0.5c_1c_2^2 \phi_1(x_1)$ if $|x_2| \le c_2$ and $|x_1| \le c_1$, Thus we can take $\gamma_{11} = 0$, $\gamma_{12} = 1.5c_1^2$, $\gamma_{21} = 0.5c_1c_2^2$, $\gamma_{22} = 0$

The S matrix is then: $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1.5c_1^2 \\ 0.5c_1c_2^2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1.5c_1^2 \\ -0.5c_1c_2^2 & 1 \end{pmatrix}$

$\det S = 1 - 0.75c_1^3c_2^2$, which can be made positive if c_1 and c_2 are small enough, eg $c_1 = c_2 = 1$. Then we can design d_1 and d_2 such that $DS + S^TD$ is positive definite.
The composite Lyapunov function is $V(x) = d_1V_1(x_1) + d_2V_2(x_2)$
The origin is locally asymptotically stable

Singular Perturbations

Consider systems of the type :
$$\dot{x} = f(t, x, z, \varepsilon)$$
$$\varepsilon \dot{z} = g(t, x, z, \varepsilon)$$

with ε very small

Example: Linear system :
$$\dot{x} = a_{11} x + a_{12} z$$
$$\varepsilon \dot{z} = -a_{22} z \qquad \text{, with } a_{22} > 0$$

$\dot{z} = -\frac{a_{22}}{\varepsilon} z$ means if ε is very small, then z is going to converge to 0

very fast. Thus there are two time scales :

Fast : $\dot{z} = -\frac{a_{22}}{\varepsilon} z \rightsquigarrow z \to 0$

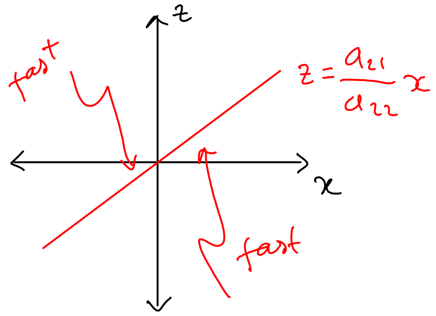
slow : $\dot{x} = a_{11} x + a_{12} \cdot 0 = a_{11} x$

Consider a different system :
$$\dot{x} = a_{11} x + a_{12} z$$
$$\varepsilon \dot{z} = a_{21} x - a_{22} z \quad \text{, with } a_{22} > 0$$

This time the fast and slow dynamics are coupled :

Fast : $\dot{z} = -\frac{a_{22}}{\varepsilon} z + \frac{a_{21}}{\varepsilon} x \to z$ converges to $\frac{a_{21}}{a_{22}} x$ very fast

Slow : $\dot{x} = a_{11} x + a_{12} z \approx a_{11} x + a_{12} \cdot \frac{a_{21}}{a_{22}} x \approx \left(a_{11} + a_{12} \frac{a_{21}}{a_{22}}\right) x$

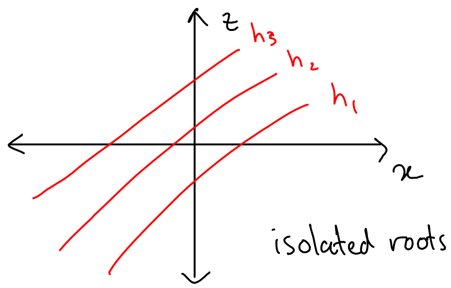


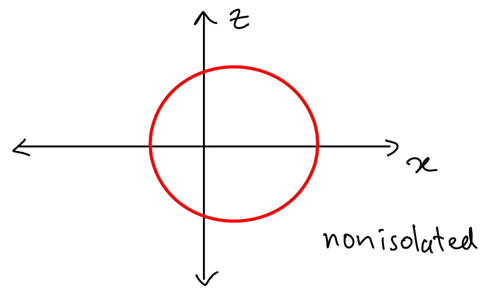
Notice the importance of the stability of the fast dynamics.

Consider the standard time invariant model:

$$\dot{x} = (x, z, \varepsilon) \quad ; \quad x(0) = x_0$$
$$\varepsilon \dot{z} = (x, z, \varepsilon) \quad ; \quad z(0) = z_0$$

Suppose that $g(x, z, 0) = 0$ has K isolated roots that let us write z in terms of x: $\quad z = h_i(x) \quad , i = 1, 2, \ldots, K$





With stability, the fast dynamics will converge to one of the branches:
$$z = h(x)$$

Slow dynamics: $\dot{x} = f(x, h(x), 0)$

Define: $y \triangleq z - h(x)$, the standard model becomes:

$$\boxed{\begin{aligned} \dot{x} &= f(x, y + h(x), \varepsilon) \\ \varepsilon \dot{y} &= \varepsilon \dot{z} - \varepsilon \frac{\partial h}{\partial x} \dot{x} = g(x, y + h(x), \varepsilon) - \varepsilon \frac{\partial h}{\partial x} f(x, y + h(x), \varepsilon) \end{aligned}}$$

New time scale: $\tau = \frac{1}{\varepsilon} t$ ("slow motion" time)

Fast dynamics: $\dfrac{dy}{d\tau} = \varepsilon \dfrac{dy}{dt} = g(x, y + h(x), \varepsilon) - \varepsilon \dfrac{\partial h}{\partial x} f(x, y + h(x), \varepsilon)$

As $\varepsilon \downarrow 0$, $\quad \boxed{\dfrac{dy}{d\tau} = g(x_0, y + h(x_0), 0) \quad ; y(0) = z_0 - h(x_0)} \quad (*)$

Notice that $y = 0$ is an equilibrium. If $(*)$ is locally exponentially stable uniformly in x (+ some smoothness conditions), the error of the approximation is $O(\varepsilon)$ (see Thm 11.1 for details)

Examples / Application:

- Rigid body dynamics in extremely low Reynolds number environment:

Newtonian mechanics: $M\ddot{x} = F - K\dot{x}$

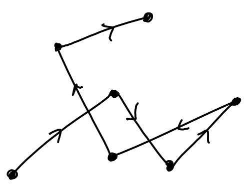
$$\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{K}{M} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{F}{M} \end{pmatrix}, \quad \frac{K}{M} \gg 1$$

Fast dynamics: $\dot{x}_2 = -\frac{K}{M} x_2 + \frac{F}{M}$

$$x_2 \to F/K$$

Slow dynamics: $\dot{x}_1 = F/K$ $\leftarrow$ Notice: first order model

instantaneous acceleration



E. coli: $F \sim 0.45 \cdot 10^{-12}$ kg m s^{-2}

$K \sim 10^{-8}$ kg s^{-1}

$M \sim 10^{-15}$ kg

- Enzymatic reactions:

$x \to *$ $\qquad \frac{dx}{dt} = -Kx$

$x + y \to z$ $\qquad \frac{dx}{dt} = -Kxy = \frac{dy}{dt}$; $\frac{dz}{dt} = Kxy$

$$E + S \underset{K_1}{\overset{K_2}{\rightleftarrows}} C \xrightarrow{K_3} E + P \quad ; K_1, K_2 \gg K_3$$

$$\frac{dE}{dt} = -K_1 ES + K_2 C + K_3 C$$

$$\frac{dS}{dt} = -K_1 ES + K_2 C$$

$$\frac{dC}{dt} = K_1 ES - (K_2 + K_3) C$$

$$\frac{dP}{dt} = K_3 C$$

Fast dynamics: $K_1 ES = K_2 C$

$C = \frac{K_1}{K_2} ES$

Slow dynamics: $\frac{dP}{dt} = \frac{K_3 \cdot K_1}{K_2} E \cdot S$

$S \xrightarrow{K} P$ with $K = \frac{K_3 K_1}{K_2} E$