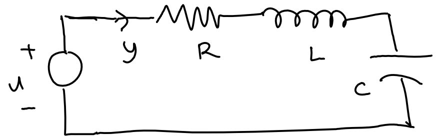
Frequency Domain Analysis (Linear Time Invariant Systems)



$$u = y \cdot R + L\frac{dy}{dt} + \frac{1}{C}\int_0^t y(\tau)\,d\tau$$

$$U(s) = Y(s)\left(R + sL + \frac{1}{sC}\right)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{Cs}{LCs^2 + RCs + 1}}$$

Proper transfer function: $G(s) = \frac{N(s)}{D(s)}$ is proper if $\deg(N(s)) \leq \deg(D(s))$

Positive real transfer functions: A $p \times p$ proper rational transfer function matrix $G(s)$ is positive real if

* poles of all elements of $G(s)$ are in $Re(s) \leq 0$
* for all $\omega \in \mathbb{R}$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G^T(-j\omega)$ is positive semidefinite

  Complex valued matrices: $M \in \mathbb{C}^{n \times n}$ is + semidefinite if for all $x \in \mathbb{C}^n$,
  $$x^* M x \geq 0$$
* any pure imaginary pole $j\omega$ is simple, and the residue matrix $\lim_{s \to j\omega} (s - j\omega) G(s)$ is positive semidefinite Hermitian.

  Note: M is Hermitian if $M^* = M$

Special case: scalar, $p = 1$: $G(s)$ is positive real if

- all the poles are in $Re(s) \leq 0$, any pole on the imaginary axis is simple
- $\lim_{s \to j\omega} (s - j\omega) G(s)$ is nonnegative real for any pole at $j\omega$
- $G(j\omega) + G^T(-j\omega) = 2 \cdot Re(G(j\omega)) \geq 0$

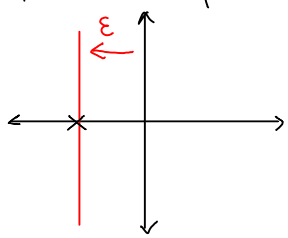
Example: $G(s) = \frac{1}{s}$ → one simple pole on the imaginary axis

The residue: $\lim_{s \to 0} s \cdot \frac{1}{s} = 1$

$Re(G(j\omega)) = Re\left(\frac{1}{j\omega}\right) = 0$, thus $G(s)$ is positive real TF

Strictly positive real: $G(s)$ is strictly positive real if $G(s-\varepsilon)$ is positive real for some $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Graphical interpretation: shifting the imaginary axis



Observe that if $G(s)$ is PR, then $G(s+\varepsilon)$ is also positive real.

Lemma: Suppose that $[G(s) + G^T(-s)]$ is not identically zero, then $G(s)$ is strictly Positive real if and only if

- $G(s)$ is Hurwitz
- $G(j\omega) + G^T(-j\omega) > 0$, $\forall \omega \in \mathbb{R}$
- $G(\infty) + G^T(\infty) > 0$ or if rank $(G(\infty) + G^T(\infty)) = q$, then

  $\lim\limits_{\omega \to \infty} \omega^2 M^T (G(j\omega) + G^T(-j\omega)) M > 0$ for any $p \times (p-q)$ full rank matrix M

  such that $M^T [G(\infty) + G^T(\infty)] M = 0$ $\left\{ \begin{array}{l} \text{the columns of } M \text{ spans the} \\ \text{kernel of } [G(\infty) + G^T(\infty)] \end{array} \right.$

Special case: scalar

$G(s)$ is strictly positive real if and only if

- $G(s)$ is Hurwitz
- $\mathrm{Re}[G(j\omega)] > 0$
- $G(\infty) > 0$ or $\lim\limits_{\omega \to \infty} \omega^2 \mathrm{Re}[G(j\omega)] > 0$

Consequence: relative degree of $G(s)$ cannot be higher than 2 !

Example: $\dfrac{1}{s+1}$ is SPR

Example: $G(s) = \begin{bmatrix} \dfrac{s+2}{s+1} & \dfrac{1}{s+2} \\ \dfrac{-1}{s+2} & \dfrac{2}{s+1} \end{bmatrix}$ is Hurwitz

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{j\omega+2}{j\omega+1} & \frac{1}{j\omega+2} \\ \frac{-1}{j\omega+2} & \frac{2}{j\omega+1} \end{bmatrix} + \begin{bmatrix} \frac{-j\omega+2}{-j\omega+1} & \frac{-1}{-j\omega+2} \\ \frac{1}{-j\omega+2} & \frac{2}{j\omega+1} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & \frac{-2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix}$$

$$\det\left(G(j\omega) + G^T(-j\omega)\right) = \frac{8(2+\omega^2)}{(1+\omega^2)^2} - \frac{4\omega^2}{(4+\omega^2)^2}$$

$$= \frac{8(2+\omega^2)(4+\omega^2)^2 - (1+\omega^2)^2 4\omega^2}{(1+\omega^2)^2 \cdot (4+\omega^2)^2} > 0$$

$$G(\infty) + G^T(\infty) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \to \text{rank} = 1 \to M = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\lim_{\omega \to \infty} \omega^2 M^T\left[G(j\omega) + G^T(-j\omega)\right]M = \lim_{\omega \to \infty} \omega^2 \cdot \frac{4}{1+\omega^2} = 4 > 0$$

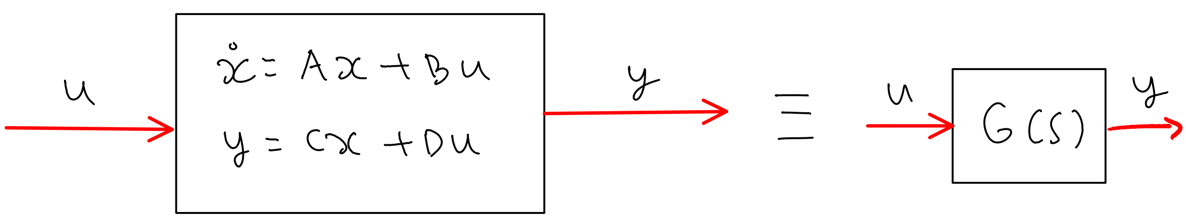
Thus, $G(s)$ is strictly positive real.

Any proper transfer function matrix $G(s)$ can be realized as:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

, where (A, B) is controllable and (A, C) is observable



(cf. minimal realization theory)

Thus: $G(s) = C(sI - A)^{-1}B + D$

PR Lemma: $G(s)$ is positive real if and only if there exist P symmetric positive definite, L and W such that:

$$\begin{aligned} PA + A^T P &= -L^T L \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

$\boxed{\text{Theorem: If } G(s) \text{ is PR, then the system is passive}}$

Proof: Use $V(x) = \frac{1}{2} x^T P x$ as storage function

$$\frac{dV}{dt} = \frac{1}{2}\left(\dot{x}^T P x + x^T P \dot{x}\right) = \frac{(x^T A^T + u^T B^T) P x + x^T P (Ax + Bu)}{2}$$

$$= \frac{1}{2} x^T (A^T P + PA) x + x^T P B u$$

$$y^T u = x^T C^T u + u^T D^T u$$

$$y^T u - \frac{dV}{dt} = x^T C^T u + u^T D^T u - \frac{1}{2} x^T (A^T P + PA) x - x^T P B u$$

$$= \cancel{x^T C^T u} + u^T \frac{(D + D^T)}{2} u - \frac{1}{2} x^T L^T L x - \cancel{x^T C^T u} + x^T L^T W u$$

$$= \frac{1}{2} u^T W^T W u - \frac{1}{2} x^T L^T L x + x^T L^T W u = \frac{1}{2} (Lx + Wu)^T (Lx + Wu)$$

$$\geq 0$$

K-Y-P Lemma: $G(s)$ is strictly positive real if and only if there exist P symmetric positive definite, L and W, and $\varepsilon > 0$ such that:

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P \\ PB &= C^T - L^T W \\ W^T W &= D + D^T \end{aligned}$$

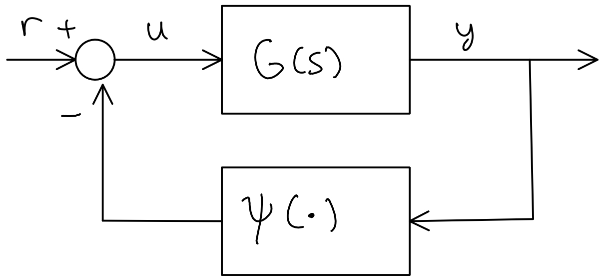
$\boxed{\text{Theorem: If } G(s) \text{ is SPR, then the system is strictly passive}}$

Proof: Similar as above, except:

$$y^T u - \dot{V} = \frac{1}{2} (Lx + Wu)^T (Lx + Wu) + \frac{1}{2} \varepsilon x^T P x$$

$$\geq \frac{1}{2} \varepsilon x^T P x$$

Linear Systems with nonlinear feedback



$$\dot{x} = Ax + Bu \qquad\qquad u = \psi(y)$$
$$y = Cx + Du$$

Lure's problem: when is the system stable for a class of nonlinear feedback?

$\psi(\cdot)$ can represent nonlinear nonidealities in control implementation

Example: $\dot{x} = Ax + Bu$

$y = x$, with stabilizing feedback control $u = -Fy$

Thus: $\dot{x} = (A - BF)x$, $(A-BF)$ is Hurwitz

consider the effect of control input saturation:

$$u_i = \varphi\big((-Fy)_i\big) \;\rightarrow\; u_i = -(Fy)_i - \underbrace{\big(\varphi(Fy)_i - (Fy)_i\big)}_{\psi(y)}$$