$\mathcal{L}_2$ and Lyapunov Stability of Passive Systems

Consider the system : $\boxed{\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}}$ $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^p$

$f: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is Lipschitz , $f(0,0) = 0$
$h: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is continuous , $h(0,0) = 0$

Lemma: If the system is output strictly passive with $y^T u - \dot{V} \geq \delta y^T y$ for some $\delta > 0$, then it is also finite gain $\mathcal{L}_2$ stable, with gain less than or equal to $1/\delta$

Proof:

$$\left(\sqrt{\tfrac{\delta}{2}}\, y - \sqrt{\tfrac{1}{2\delta}}\, u\right)^T \left(\sqrt{\tfrac{\delta}{2}}\, y - \sqrt{\tfrac{1}{2\delta}}\, u\right) = \frac{\delta}{2} y^T y - y^T u + \frac{1}{2\delta} u^T u$$

$$y^T u = -\left(\sqrt{\tfrac{\delta}{2}}\, y - \sqrt{\tfrac{1}{2\delta}}\, u\right)^T \left(\sqrt{\tfrac{\delta}{2}}\, y - \sqrt{\tfrac{1}{2\delta}}\, u\right) + \frac{\delta}{2} y^T y + \frac{1}{2\delta} u^T u$$

$$\delta y^T y \leq y^T u - \dot{V} \leq \frac{\delta}{2} y^T y + \frac{1}{2\delta} u^T u - \dot{V}$$

$$\frac{\delta}{2} y^T y \leq \frac{1}{2\delta} u^T u - \dot{V} \Rightarrow \frac{\delta}{2} \|y\|_{\mathcal{L}_2}^2 \leq \frac{1}{2\delta} \|u\|_{\mathcal{L}_2}^2 + V(0) - V(\infty)$$

$$\leq \frac{1}{2\delta} \|u\|_{\mathcal{L}_2}^2 + V(0)$$

$$\|y\|_{\mathcal{L}_2}^2 \leq \frac{1}{\delta^2} \|u\|_{\mathcal{L}_2}^2 + \frac{2}{\delta} V(0) \leq \frac{1}{\delta^2} \|u\|_{\mathcal{L}_2}^2 + \frac{2}{\delta} V(0) + 2 \cdot \frac{1}{\delta} \|u\| \sqrt{\frac{2}{\delta} V(0)}$$

$$\|y\|_{\mathcal{L}_2} \leq \frac{1}{\delta} \|u\|_{\mathcal{L}_2} + \sqrt{\frac{2}{\delta} V(0)}$$

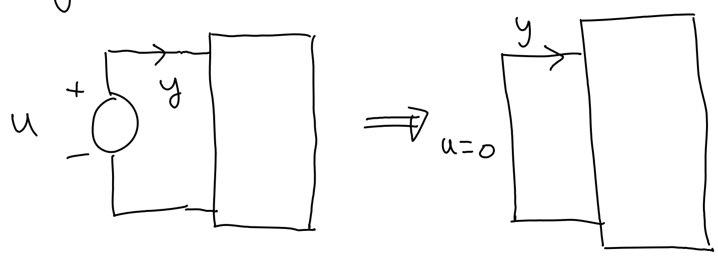
Note: compare with Thm 5.5

Lyapunov Stability

- If the system is passive with + def storage function, then the unforced system is stable.

  Proof: Use the storage function as Lyapunov function, then if $u=0$,

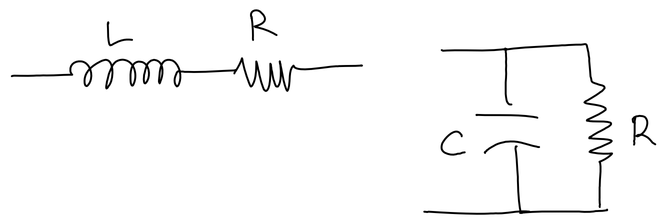
  $y^T u - \dot{V} \geq 0 \implies \dot{V} \leq 0$



- If the system is strictly passive: $y^T u - \dot{V} \geq \psi(x)$, ψ is + def then the unforced system is asymptotically stable

  Proof: $\dot{V} \leq \psi(x) \rightarrow$ converges to $\{x \mid \psi(x) = 0\}$, which is $x=0$

  Circuit interpretation:

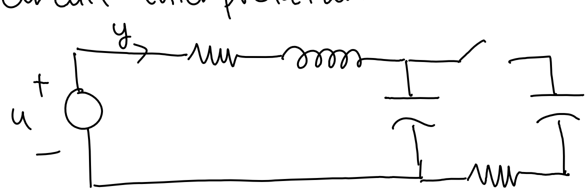


- If the system is output-strictly passive and zero state observable then the unforced system is asymptotically stable

  Zero state observable: $y(t) \equiv 0 \implies x(t) \equiv 0$

  Proof: $\dot{V} \leq -y^T \rho(y) \rightarrow$ converges to $\{x \mid h(x,0) = 0\}$, which is $x=0$

  Circuit interpretation:



← output strictly passive but not zero state observable

Example: $\dot{x}_1 = x_2$

$$\dot{x}_2 = -a x_1^3 - k x_2 + u$$

$$y = x_2$$

$a, k > 0$. Consider the storage function: $V(x) = \frac{1}{4} a x_1^4 + \frac{1}{2} x_2^2$

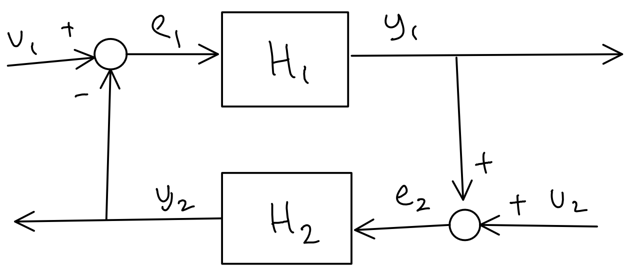
$$\dot{V} = a x_1^3 \cdot \dot{x}_1 + x_2 \cdot \dot{x}_2 = \cancel{a x_1^3 x_2} - \cancel{a x_1^3} x_2 - k x_2^2 + u x_2$$

$y^T u - \dot{V} = k x_2^2 = k y^2$ $\Leftarrow$ output strictly passive

$y(t) \equiv 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow -a x_1^3 \equiv 0 \Rightarrow x_1(t) \equiv 0$

thus zero-state observable. The unforced system is asymp. stable.

Feedback systems



$$\dot{x}_1 = f_1(x_1, e_1) \qquad \dot{x}_2 = f_2(x_2, e_2)$$
$$y_1 = h_1(x_1, e_1) \qquad y_2 = h_2(x_2, e_2)$$

Thm: If both H_1 and H_2 are passive, then the feedback system is passive

Proof: Let $V_1(x)$ and $V_2(x)$ be the storage functions of H_1 and H_2.

$$y_1^T e_1 - \dot{V}_1 \geq 0 \quad \text{and} \quad y_2^T e_2 - \dot{V}_2 \geq 0$$

Take $V(x) = V_1(x) + V_2(x)$

$$y_1^T u_1 + y_2^T u_2 = y_1^T (e_1 + y_2) + y_2^T (e_2 - y_1) = y_1^T e_1 + y_2^T e_2$$

$$\geq 0$$

Similarly: strictly passive, output-strictly passive

Lemma: If H_1 and H_2 are output strictly passive, then the feedback system is finite-gain $\mathcal{L}_2$ stable.

Note: we bypass small-gain theorem.

Thm: Suppose that:

$$y_1^T e_1 - \dot{V}_1 \geq \varepsilon_1 e_1^T e_1 + \delta_1 y_1^T y_1$$
$$y_2^T e_2 - \dot{V}_2 \geq \varepsilon_2 e_2^T e_2 + \delta_2 y_2^T y_2$$

The closed loop system is finite gain $\mathcal{L}_2$ stable if:

$$\varepsilon_1 + \delta_2 > 0$$
$$\varepsilon_2 + \delta_1 > 0$$

Notice that $\varepsilon_{1,2}$ and $\delta_{1,2}$ can be negative

Lyapunov stability can follow from (strict) passivity as discussed previously:

* If both systems are:
  - passive, then the unforced feedback system is stable
  - strictly passive, then the unforced feedback system is asymp. stable
  - output strictly passive and zero state observable, then the unforced feedback system is asymp. stable

For the last bullet, we need to show that if both systems are zero state observable, then the feedback system is also zero state observable.

$$(y_1 \equiv 0) \,\&\, (y_2 \equiv 0) \Rightarrow (e_1 \equiv 0) \,\&\, (e_2 \equiv 0) \Rightarrow (x_1 \equiv 0) \,\&\, (x_2 \equiv 0)$$