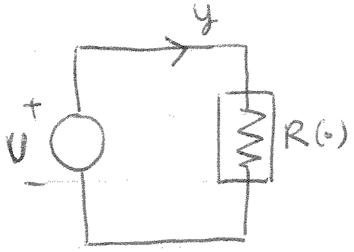
Passivity

Resistive elements in electric circuits:



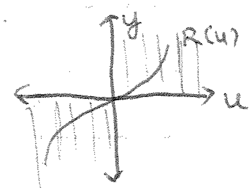
$y = R(u)$

Instantaneous power absorbed by the element:

$W(t) = y(t) \cdot u(t) \rightarrow$ ⊕ absorbs/dissipates energy

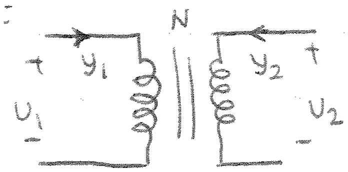
⊖ generates energy

For passive elements: $W(t) \geq 0$: the u-y relation lies on the first and third quadrants.



Special case: $y(t) \cdot u(t) = 0 \rightarrow$ lossless system

Example: Lossless transformer:

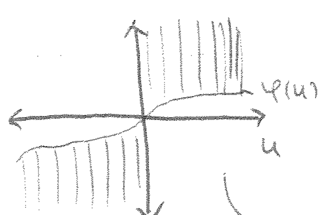


$$u_2 = N \cdot u_1 \qquad \rightarrow \qquad y^T u = y_1 u_1 + y_2 u_2$$
$$y_2 = -\frac{1}{N} y_1 \qquad\qquad = y_1 u_1 - \frac{1}{N} y_1 \cdot N u_1 = 0$$

Input strict passivity: Suppose that there exists ~~an~~ a $\varphi(u)$ such that

$$y^T u \geq \varphi^T(u) \cdot u$$

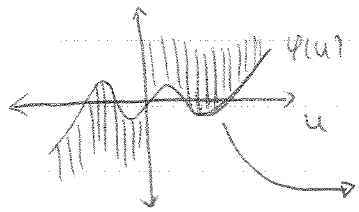
If $\varphi^T_{(u)} u > 0$, then the system is called input-strictly-passive



$\rightarrow y^T u = 0$ only if $u = 0$

excess of ~~storage~~ passivity

If $\varphi^T(u)u$ is not positive definite then:

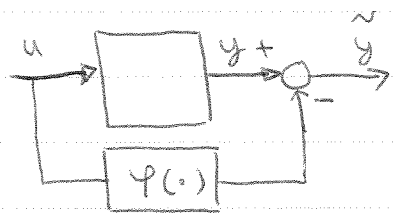


Shortage of ~~storage~~ passivity

can be made passive in feedforward operation:

Define: $\tilde{y} = y - \varphi(u)$: $\tilde{y}^T u = y^T u - u^T \varphi(u) \geq 0$

The system defined in terms of the new input-output pair $(u, \tilde{y})$ is passive



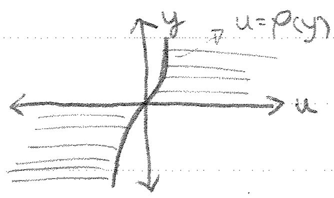
not input-strictly-passive but input-feedforward-passive

Similarly, we can define:

Output Strict Passivity

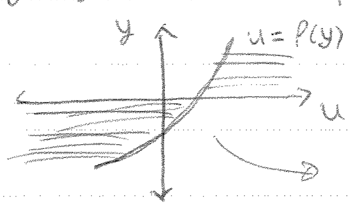
Suppose that there exists a $\rho(y)$ such that: $y^T u \geq y^T \rho(y)$

If $y^T \rho(y)$ is + definite, then the system is output strictly passive



← excess of passivity

If $y^T \rho(y)$ is not + definite, then:



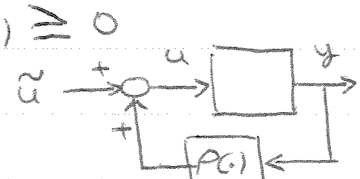
shortage of passivity

can be made passive in feedback-operation:

$\tilde{u} = u - \rho(y) \rightarrow y^T \tilde{u} = y^T u - y^T \rho(y) \geq 0$

$\downarrow$

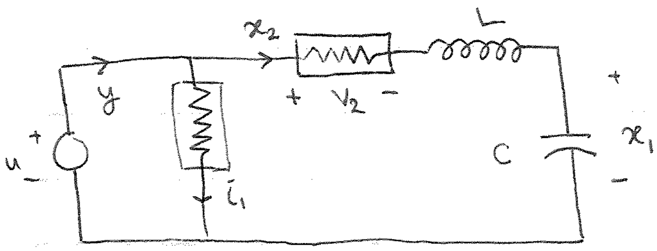
$u = \tilde{u} + \rho(y)$



Passivity can be defined not only on electric circuits. Input-output pairs that are <u>power pairs</u> can be treated similarly.

Examples of power pairs: voltage - current, force - velocity, torque - angular vel. (hydrolic) pressure - debit, etc

Passivity of dynamical systems



the circuit has passive elements that can store energy

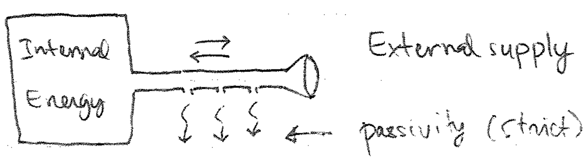
$$\boxed{\begin{aligned} i_1 &= h_1(u) \\ V_2 &= h_2(x_2) \end{aligned}}$$

Internal energy storage $= V(x) = \frac{1}{2} L x_2^2 + \frac{1}{2} C x_1^2$

External power: $y \cdot u$, External energy supplied up to time $t = \int_0^t y(\tau) \cdot u(\tau)\, d\tau$

The system is passive if $\boxed{V(x(t)) - V(x(0)) \le \int_0^t y(\tau)\, u(\tau)\, d\tau \quad \text{for all } t \ge 0}$

Illustration:



Equivalently: $\boxed{\frac{dV}{dt} \le y(t) \cdot u(t)}$

Check the example: $\frac{dV}{dt} = L x_2 \dot{x}_2 + C x_1 \dot{x}_1$

$$= \cancel{L} x_2 \cdot \frac{1}{\cancel{L}} (u - h_2(x_2) - x_1) + \cancel{C} x_1 \cdot \frac{1}{\cancel{C}} x_2$$

$$= x_2 u - x_2 h_2(x_2)$$

$y \cdot u = (i_1 + x_2) \cdot u = h_1(u) \cdot u + x \cdot u$, thus

$$y \cdot u - \frac{dV}{dt} = h_1(u) u + h_2(x_2) x_2$$

If both $h_1(\cdot)$ and $h_2(\cdot)$ belong to sector $[0,\infty]$ then the system is passive.

$\hookrightarrow$ sector $[a,b]$

Suppose that only $h_2 \in [0,\infty]$, then:

$$y \cdot u - \frac{dv}{dt} \geq h_1(u) \cdot u$$

$$(y - h_1(u))u - \frac{dv}{dt} \geq 0$$, thus if we redefine the output as:

$\tilde{y} \triangleq y - h_1(u)$, the system is passive. Notice that $\tilde{y} = x_2$, thus this can be interpreted as removing the "nonpassive" element out of the system

General case: $\boxed{\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned}}$

Definition: the system is passive if there exists a continuosly differentiable positive semidefinite function $V(x)$ such that

$$y^T u \geq \frac{dv}{dt} = \frac{\partial V}{\partial x} f(x,u)$$

$V(x)$ is called the storage function, and can be interpreted as potential energy.

Similarly: $y^T u = \dot{V} \rightarrow$ lossless

$y^T u \geq \dot{V} + u^T \varphi(u)$ $\rightarrow$ input strictly passive if $u^T \varphi(u)$ is + def

$\rightarrow$ input - feedforward passive otherwise

$\vdots$

etc

Example: $\dot{x} = u$ is passive / lossless w.r.t storage function $V(x) = \frac{1}{2}x^2$

$y = x$

$\frac{dV}{dt} = x \cdot \dot{x} = y \cdot u$. Notice that unlike Lyapunov functions, we <u>cannot</u> scale storage function. Take for example: $\tilde{V}(x) = x^2$

$$\frac{d\tilde{V}}{dt} = 2 \cdot x \cdot \dot{x} = 2 \cdot y \cdot u \neq y \cdot u$$

Also notice that the I/O system is <u>not</u> finite gain $\mathcal{L}_p$ stable.