Consider a system: $\dot{x} = f(x,u,t)$, $y = h(x,u,t)$ $\Big\}$ $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$

$$f(0,0,t) = 0, \ \forall t \geq 0$$

Suppose that:

- $\dot{x} = f(x,u,t)$ is ISS
- h satisfies: $\|h(x,u,t)\| \leq \alpha_1(\|x\|) + \alpha_2(\|u\|) + \eta$ globally, α_1, α_2 are class K functions, and η is a positive constant

Thm (5.3): For each $x_0 \in \mathbb{R}^n$, the I/O system is L_∞ stable

Example: $\dot{x} = -x - 2x^3 + (1+x^2)u^2$

$y = x^2 + u$

Check for ISS: Define $V(x) = x^2$,

$$\begin{aligned} \dot{V}(x) &= x(-x - 2x^3 + (1+x^2)u^2) \\ &= -x^2 - 2x^4 + (x+x^3)u^2 \\ &= -x^4 - (x^2 - xu^2) - (x^4 - x^3u^2) \\ &\leq -x^4 \quad \text{if } |x| \geq u^2 \end{aligned}$$

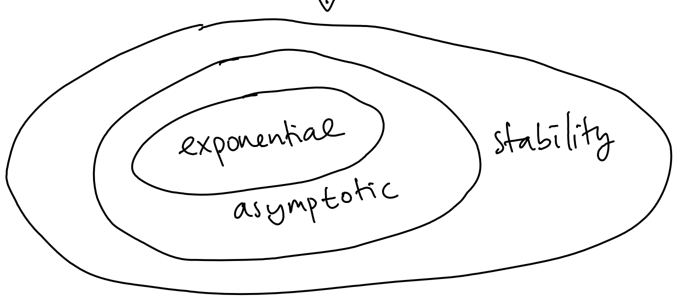
Therefore: $\dot{x} = -x - 2x^3 + (1+x^2)u^2$ is ISS. Moreover:

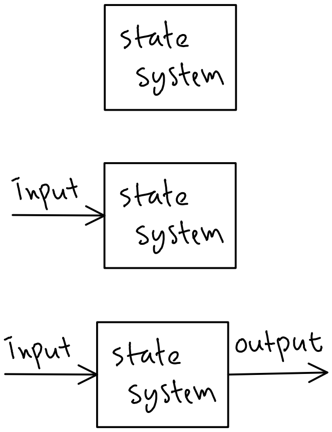
$$|x^2 + u| \leq \underbrace{|x|^2}_{} + \underbrace{|u|}_{}$$

Class K functions

Thus the I/O system is L_∞ stable (not finite gain L_∞ stable though!)

RECAP: Stability of state systems $\rightarrow$ generic
$\rightarrow$ uniform (TV systems)





Exponential stability + smoothness + Lipschitz (Ch. 4)

$\Downarrow$ input enters "nicely"

Input to state Stability (Ch 4.8)

$\Downarrow$ output behaves "nicely"

Finite gain $\mathcal{L}_p$ stability (Ch. 5)

$\mathcal{L}_2$ gain

In some cases, we also want to compute the I/O gain, in addition to establishing stability. In particular $\mathcal{L}_2$ gain is popular because of the energy interpretation. Recall that $\mathcal{L}_2$ gain for linear systems corresponds to H_∞ norm of the transfer function (frequency domain)

Thm: Given a system: $\dot{x} = f(x) + G(x)u$; $f(0) = 0$

$y = h(x)$; $h(0) = 0$

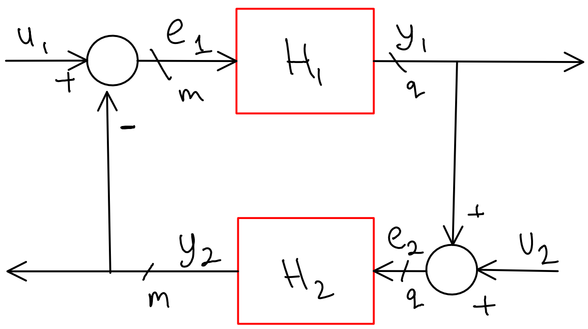
$f(x)$ is locally Lipschitz, $G(x)$ and $h(x)$ continuous. Suppose that $V(x)$ is continuously differentiable and positive semidefinite and

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

for all $x \in \mathbb{R}^n$, where γ is a pos. constant. Then, the system is finite gain $\mathcal{L}_2$ stable with gain less than or equal to γ.

The inequality is called Hamilton-Jacobi inequality.
Note that $V(x)$ needs not be pos. definite. Interpretation of linear systems: controllability and observability.

Feedback systems : small gain theorem



Suppose that both H_1 and H_2 are finite gain L_p stable :

$$\|y_{1\tau}\| \le \gamma_1 \|e_{1\tau}\| + \beta_1 \quad , \forall e_1 \in \mathcal{L}_{pe}^m \quad , \tau \ge 0$$
$$\|y_{2\tau}\| \le \gamma_2 \|e_{2\tau}\| + \beta_2 \quad , \forall e_2 \in \mathcal{L}_{pe}^q \quad , \tau \ge 0$$

Also, assume that the system is well-posed, i.e. for any $u_1 \in \mathcal{L}_{pe}^m$ and $u_2 \in \mathcal{L}_{pe}^q$, e_1, e_2, y_1, and y_2 are well defined. (A sufficient condition for this could use the contraction mapping thm)

Thm: The feedback system (seen as I/O system from (u_1, u_2) to (y_1, y_2)) is finite gain $\mathcal{L}_p$ stable if $\gamma_1\gamma_2 < 1$.

Proof: $e_1 = u_1 - y_2$

$e_2 = u_2 + y_1$

$$\left.\begin{aligned} y_1 &= H_1(e_1) = H_1(u_1 - y_2) \\ y_2 &= H_2(e_2) = H_2(u_2 + y_1) \end{aligned}\right\} \quad \begin{aligned} \|y_1\| &\le \gamma_1 \|u_1\| + \gamma_1 \|y_2\| + \beta_1 \\ &\le \gamma_1 \|u_1\| + \gamma_1 (\gamma_2 \|u_2\| + \gamma_2 \|y_1\| + \beta_2) + \beta_1 \end{aligned}$$

$$\|y_1\| \le \frac{\gamma_1 \|u_1\| + \gamma_1\gamma_2 \|u_2\| + \beta_1 + \gamma_1\beta_2}{1 - \gamma_1\gamma_2}$$

Similarly, we can obtain : $\|y_2\| \le \dfrac{\gamma_2 \|u_2\| + \gamma_1\gamma_2 \|u_1\| + \beta_2 + \gamma_2\beta_1}{1 - \gamma_1\gamma_2}$

$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, thus $\|y\| \leq \|y_1\| + \|y_2\|$. From the previous inequalities, we establish $\mathcal{L}_p$ finite gain stability.

Interpretation: u_1 and u_2 are disturbance signals. Designing $\mathcal{L}_p$ finite-gain stable feedback loop ensures that the effect of the disturbance is "manageable". See Example (5.14).