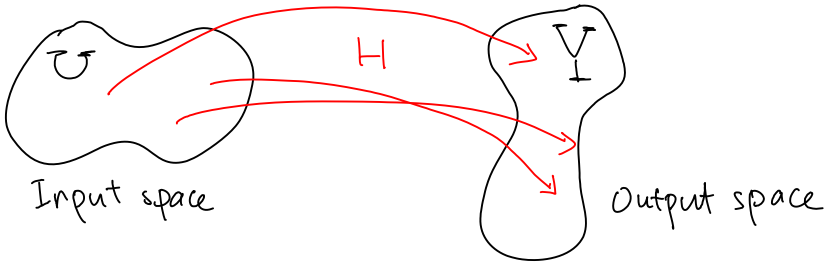
Input - Output Stability

Systems as input-output relation / map :



$y = H(u)$, $y \in Y$, $u \in U$,

Y and U are normed function spaces

$\mathcal{L}_p$ spaces

The $\|\cdot\|_{\mathcal{L}_p}$ norm of a function $u : [0,\infty) \to \mathbb{R}^m$ is defined as

$$\boxed{\|u\|_{\mathcal{L}_p} \triangleq \left(\int_0^\infty \|u(\tau)\|^p d\tau \right)^{1/p}}$$

If $\|u\|_{\mathcal{L}_p} < \infty$, then $u \in \mathcal{L}_p^m$, for $p \in [1,\infty)$.

$$\boxed{\|u\|_{\mathcal{L}_\infty} \triangleq \sup_{\tau \geq 0} \|u(\tau)\|}$$

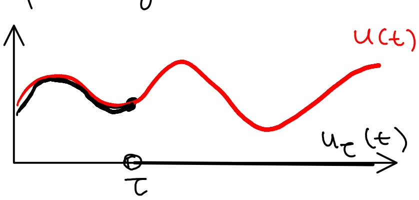
Example : $u(t) = \begin{cases} 0 & , t < 1 \\ 1/t & , t \geq 1 \end{cases}$

$$\left.\begin{aligned} \|u\|_{\mathcal{L}_1} &= \int_1^\infty \frac{1}{t} dt = \infty \\ \|u\|_{\mathcal{L}_2} &= \int_1^\infty \frac{1}{t^2} dt = -\frac{1}{t}\Big|_1^\infty = 1 \end{aligned}\right\} \; u \in \mathcal{L}_2 \text{ but } u \notin \mathcal{L}_1$$

Extended $\mathcal{L}_p$ spaces

Truncation : u_τ is a truncation of u defined by :

$u_\tau(t) = \begin{cases} u(t) , & 0 \leq t \leq \tau \\ 0 & , t > \tau \end{cases}$



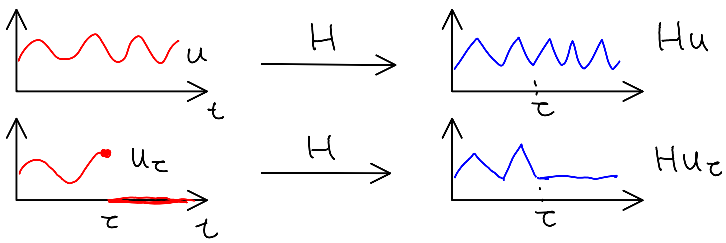
$u \in \mathcal{L}_{pe}^m$ if $u_\tau \in \mathcal{L}_p^m$ for all $\tau \in [0,\infty)$

Example: $u(t) = \begin{cases} 1/t & , t \geq 1 \\ 0 & , t < 1 \end{cases}$

$\|u_\tau\|_{\mathcal{L}_1} = \int_1^\tau 1/t \, dt = \ln \tau < \infty \quad , \forall \tau \in [0,\infty)$ thus $u \in \mathcal{L}_{1e}$

Causal systems / mappings: A mapping $H: \mathcal{L}_{pe}^m \to \mathcal{L}_{pe}^q$ is causal if

$$(Hu)_\tau = (Hu_\tau)_\tau$$



Stable systems / mappings: A mapping $H: \mathcal{L}_{pe}^m \to \mathcal{L}_{pe}^q$ is $\mathcal{L}_p$ stable if there exist a class K function α, defined on $[0,\infty)$, and a nonnegative constant β such that

$$\|(Hu)_\tau\|_{\mathcal{L}_p} \leq \alpha(\|u_\tau\|_{\mathcal{L}_p}) + \beta$$

for all $u \in \mathcal{L}_{pe}^m$ and $\tau \in [0,\infty)$.

H is finite gain $\mathcal{L}_p$ stable if there exist nonnegative constants γ and β such that

$$\|(Hu)_\tau\|_{\mathcal{L}_p} \leq \gamma \cdot \|u_\tau\|_{\mathcal{L}_p} + \beta \quad \cdots\cdots\cdots\cdots (*)$$

for all $u \in \mathcal{L}_{pe}^m$ and $\tau \in [0,\infty)$.

The smallest γ such that $(*)$ holds is called the gain of H.

Remark: Finite-gain L_p stable $\Rightarrow$ L_p stable

Example: $y(t) = \int_0^t h(t-s)\, u(s)\, ds$, suppose that

$\|h\|_{\mathcal{L}_1} = \int_0^\infty |h(s)|\, ds < \infty$, then, for $\forall u \in \mathcal{L}_{\infty e}$

$$|y(t)| \leq \int_0^t |h(t-s)||u(s)|\,ds \leq \int_0^t |h(t-s)|\,ds \cdot \|u\|_{\mathcal{L}_\infty}$$

$$\leq \|h\|_{\mathcal{L}_1} \cdot \|u\|_{\mathcal{L}_\infty}$$

$\mathcal{L}$ stability of state models

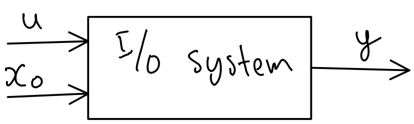
Consider systems of the form: $\boxed{\begin{aligned} \dot{x} &= f(x,u,t) \\ y &= h(x,u,t) \end{aligned}}$, where: $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^q$

$f: \mathbb{R}^n \times \mathbb{R}^m \times [0,\infty) \to \mathbb{R}^n$ is piecewise continuous in t, and Lipschitz in (x,u). Moreover: $f(0,0,t) = 0$, and

$$\|f(x,u,t) - f(x,0,t)\| \leq L\|u\| \quad , \quad L \geq 0$$

$h: \mathbb{R}^n \times \mathbb{R}^m \times [0,\infty) \to \mathbb{R}^q$ is piecewise continuous in t and continuous in (x,u). Moreover: $h(0,0,t) = 0$ and

$$h(x,u,t) \leq \eta_1\|x\| + \eta_2\|u\| \quad , \quad \eta_1, \eta_2 \geq 0$$



Suppose that there exists a Lyapunov function for the unforced system, $V(x,t)$ such that:

$$c_1\|x\|^2 \leq V(x,t) \leq c_2\|x\|^2$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,0,t) \leq -c_3\|x\|^2$$

$$\left\|\frac{\partial V}{\partial x}\right\| \leq c_4\|x\|$$

for some positive constants c_1, c_2, c_3, c_4.

Notice that this implies that the origin is globally exp. stable for the unforced system. However, it also implies that the I/O system is finite gain $\mathcal{L}_\infty$ stable.

To see this, recall the ISS result that

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0) + K \cdot \|u\|_{\mathcal{L}_\infty}$$

$$\leq \beta(\|x_0\|, 0) + K\|u\|_{\mathcal{L}_\infty} \quad \text{, thus}$$

$$\begin{aligned} \|y(t)\| &\leq \eta_1 \cdot \left(\beta(\|x_0\|,0) + K\|u\|_{\mathcal{L}_\infty}\right) + \eta_2 \cdot \|u\|_{\mathcal{L}_\infty} \\ &\leq \eta_1 \cdot \beta(\|x_0\|,0) + (\eta_1 \cdot K + \eta_2)\|u\|_{\mathcal{L}_\infty} \end{aligned}$$

For other finite gain L_p stability, we need to use the exponential convergence property.

Weaker results: If all the conditions are only satisfied locally for $x \in D \subset \mathbb{R}^n$, and $u \in D_u \in \mathbb{R}^m$, where $0 \in D$ and $0 \in D_u$, we get a weaker version of finite gain L_p stability, which is small-signal finite gain L_p stability

Def: A mapping $H: \mathcal{L}_{pe}^m \to \mathcal{L}_{pe}^q$ is small signal finite gain L_p stable if $\exists r > 0$ such that

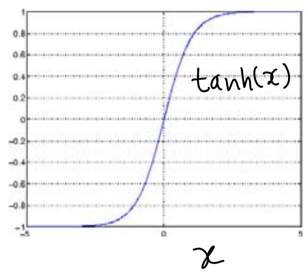
$$\|(Hu)_\tau\|_{\mathcal{L}_p} \leq \gamma \cdot \|u_\tau\|_{\mathcal{L}_p} + \beta \quad , \gamma, \beta \geq 0$$

if $\sup_{0 \leq t \leq \tau} \|u(t)\| < r$.

Example: $\dot{x} = -x - x^3 + u \ , \ x(0) = x_0$

$y = \tanh x + u$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$



Take $V(x) = \frac{1}{2}x^2 \to \frac{\partial V}{\partial x} f(x,0,t) = -x^2 - x^4 < -x^2$

$$\left\|\frac{\partial V}{\partial x}\right\| = |x|$$

$|y| \leq |\tanh x| + |u| \leq k \cdot |x| + |u|$

Thus, the I/O system is finite gain L_p stable for $p \in [1, \infty]$

Remark: If f is continuously differentiable and $\|\frac{\partial f}{\partial x}\|$ is bounded uniformly in t, finite gain L_p stability can be obtained from global exponential stability directly (without finding a Lyapunov function)