Input-to-State Stability (ISS)

Systems with input : $\xrightarrow{u} \boxed{\begin{array}{l} \dot{x} = f(x,u,t) \\ x(t_0) = x_0 \end{array}}$ $\quad x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$

f is piecewise continuous in t and locally Lipschitz in x and u.
The input signal is piecewise continuous and bounded :

$$\boxed{\sup_{t \geq t_0} \|u(t)\| = M < \infty}$$

Suppose that 0 is an equilibrium for $u=0$, i.e. $f(0,0,t)=0$, $\forall t \geq t_0$. How does u affect x ?

Consider the linear case: $\dot{x} = Ax + Bu$, with A Hurwitz :

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau)\, d\tau$$

Since A is Hurwitz, $\exists\ k, \lambda > 0$ s.t : $\|e^{A(t-t_0)}\| < k e^{-\lambda(t-t_0)}$

$$\|x(t)\| \leq k e^{-\lambda(t-t_0)} \|x_0\| + \int_{t_0}^{t} k e^{-\lambda(t-t_0)} \|B\| \cdot M \, d\tau$$

$$\leq k e^{-\lambda(t-t_0)} \|x_0\| + \frac{k\|B\|}{\lambda} M$$

Notice that : 1) With zero input ($M=0$) the origin is asymp. stable
2) the zero-state response ($\|x_0\|=0$) is bounded if the input is bounded

For nonlinear systems, these two stability properties do not necessarily overlap.

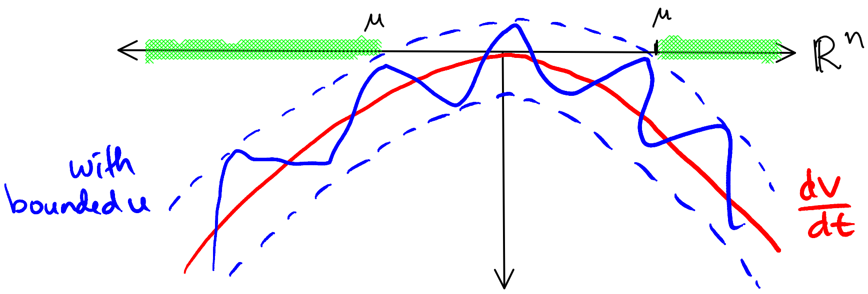
Example: $\dot{x} = -x + (x + x^2)u$

If $u(t) \equiv 0$, the origin is asymp. stable, while if we take, e.g. $u(t)=1$, the state is not bounded (and has finite escape time).

Consider the unforced system $(u \equiv 0)$: $\dot{x} = f(x, 0, t) \triangleq \hat{f}(x, t)$

Suppose that we have a Lyapunov function $V(x,t)$ that shows that 0 is uniformly asymp. stable, thus:

$$\frac{dV}{dt} \leq -\alpha(\|x\|) \quad \text{, for some class K function } \alpha$$



With bounded input, there might be an $\mu > 0$ such that:

$\frac{dV}{dt} \leq 0$ if $\|x\| > \mu$, which somehow implies that $x(t)$ is bounded

Physical interpretation:

state ~ energy

input ~ power (external)

stability ~ dissipative system with rate of energy loss ~ energy

Bounded external power results in bounded steady state energy

Definition: The system $\dot{x} = f(x, u, t)$ is input-to-state stable (ISS) if there exist a class KL function β and a class K function γ such that for any $x(t_0) \in \mathbb{R}^n$ and bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$x(t) \leq \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|\right)$$

Interpretation:

- with zero input, the origin is uniformly asymp. stable
- if $\|u(t)\|$ is kept small enough, there is a graded response from $\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|$ to $\sup_{t_0 \leq \tau \leq t} \|x(\tau)\|$

Theorem (4.18 + 4.19): Let $V : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ be a continuously differentiable function such that:

$$\alpha_1(\|x\|) \le V(x,t) \le \alpha_2(\|x\|)$$

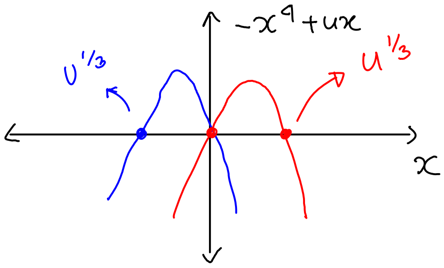
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(x,u,t) \le -W_3(\|x\|), \quad \forall \|x\| \ge \rho(\|u\|) > 0,$$

where α_1, α_2 are class K_∞ functions, W_3 is a continuous + def function, and ρ is a class K function. Then the system $\dot{x} = f(x,u,t)$ is ISS with $\gamma = a_1^{-1} \circ a_2 \circ \rho$

Example: $\dot{x} = -x^3 + u$

Define: $V(x) = \frac{1}{2}x^2$ (We can take $\alpha_1 = \alpha_2 = V$)

$$\frac{dV}{dt} = x(-x^3 + u) = -x^4 + ux$$



$$\frac{dV}{dt} = -(1-\theta)x^4 - \theta x^4 + xu \le -(1-\theta)x^4, \quad \forall |x| \ge \left(\frac{|u|}{\theta}\right)^{1/3}$$

with $0 < \theta < 1$. Thus the system is ISS with $\gamma(r) = (r/\theta)^{1/3}$

Lemma (4.6): If the unforced system is globally exponentially stable at the origin, and f is continuously differentiable and globally Lipschitz in (x,u) uniformly in t, then the system $\dot{x} = f(x,u,t)$ is ISS.

Proof: A converse theorem (4.14) guarantees the existence of a Lyapunov function $V(x,t)$ such that:

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(x,0,t) \le -c_1 \|x\|^2, \text{ and}$$

$$\left\| \frac{\partial V}{\partial x} \right\| \le c_2 \|x\|$$

Therefore: $$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(x,u,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,0,t) - \frac{\partial V}{\partial x} f(x,0,t) + \frac{\partial V}{\partial x} f(x,u,t)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,u,t) \leq -c_1 \|x\|^2 + c_2\|x\| \cdot \underset{\nwarrow \text{ Lipschitz constant}}{L} \cdot \|u\|$$

$$\downarrow \; 0 < \theta < 1$$

$$\leq -(1-\theta)c_1 \|x\|^2 \underbrace{- \theta c_1 \|x\|^2 + c_2 \|x\| L \|u\|}_{\leq 0 \text{ if } \|x\| \geq \frac{c_2 L}{c_1 \theta} \|u\|}$$

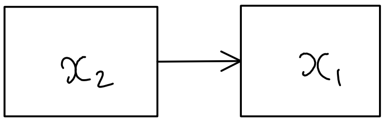
$$\leq -(1-\theta) c_1 \|x\|^2 \;, \forall \|x\| \geq \frac{c_2 L}{c_1 \theta} \|u\|$$

Note: Global Lipschitz property is important! Look at $\dot{x} = -x + (x + x^2) u$
Exponential stability is important!

Cascade systems

$\dot{x}_1 = f_1(x_1, x_2, t)$

$\dot{x}_2 = f_2(x_2, t)$



$f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times [0, \infty) \to \mathbb{R}^{n_1}$
$f_2 : \mathbb{R}^{n_2} \times [0, \infty) \to \mathbb{R}^{n_2}$
} piecewise continuous & locally Lipschitz

Suppose that both $f_1(x_1, 0, t)$ and $f_2(x_2, t)$ are globally uniformly asymp. stable, when is the cascade system uniformly asymp. stable as well?

Lemma: If both $f_1(x_1, 0, t)$ and $f_2(x_2, t)$ are globally uniformly asymp. stable, and $\dot{x}_1 = f_1(x_1, x_2, t)$ is ISS w.r.t input x_2, then the cascade system is globally uniformly asymp. stable too.