Stability of LTV systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}$$

State trajectory: $x(t) = \Phi(t,t_0)\, x(t_0)$, where:

$\phi(t,t) = I$, and $\dfrac{\partial \phi(t,t_0)}{\partial t} = A(t)\, \Phi(t,t_0)$

For LTV systems, uniform asymptotic stability is equivalent to exponential stability.

Thm: The origin is uniformly asymp. stable iff

$$\|\Phi(t,t_0)\| \le k e^{-\lambda(t-t_0)}, \quad \forall t \ge t_0$$

for some $k, \lambda > 0$

Notice that the stability property is automatically global

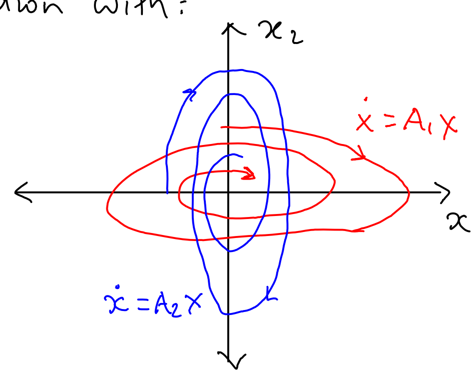
Important: In checking for stability of LTV, instantaneous stability of $A(t)$ is not sufficient (or necessary)

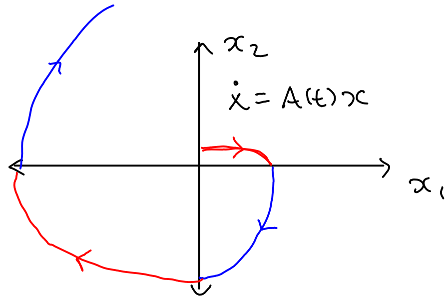
Example: $A_1 = \begin{bmatrix} -0.1 & 10 \\ -0.1 & -0.1 \end{bmatrix}$; $A_2 = \begin{bmatrix} -0.1 & 0.1 \\ -10 & -0.1 \end{bmatrix}$

$\downarrow$ $\lambda_{1,2} = -0.1 \pm j$ $\qquad$ $\downarrow$ $\lambda_{1,2} = -0.1 \pm j$

Define $A(t)$ as a 2π-periodic function with:

$$A(t) = \begin{cases} A_1, & 0 \le t < \pi/2 \\ A_2, & \pi/2 \le t < \pi \\ A_1, & \pi \le t < 3\pi/2 \\ A_2, & 3\pi/2 \le t < 2\pi \end{cases}$$





Some initial states result in unbounded trajectories $\Rightarrow$ UNSTABLE

Theorem (4.12): Suppose that the origin is uniformly asymp. stable, and $A(t)$ is continuous and bounded. Let $Q(t)$ be continuous, symmetric, and $\exists\, C_3, C_4 > 0$ such that:

$$C_3 I \leq Q(t) \leq C_4 \cdot I$$

Then, there exists a continuosly differentiable, bounded, symmetric, positive definite matrix $P(t)$ such that

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t)$$

Interpretation: $V(x,t) = x^T P(t)\, x$ is a Lyapunov function.

Proof: Take $P(t) = \int_t^\infty \Phi^T(\tau,t)\, Q(\tau)\, \Phi(\tau,t)\, d\tau$

Thus $P(t)$ is cont. differentiable and symmetric. For any $x_0 \in \mathbb{R}^n$,

$$x_0^T P(t) x_0 = \int_t^\infty x_0^T\, \Phi(\tau,t)\, Q(\tau)\, \Phi(\tau,t)\, x_0\, d\tau$$

$$= \int_t^\infty x^T(\tau)\, Q(\tau)\, x(\tau)\, d\tau$$

$$\boxed{x(\tau) \text{ satisfies } \frac{dx(\tau)}{d\tau} = A(\tau)x(\tau),\ x(t) = x_0,\ \forall\, \tau \geq t}$$

$$\leq \int_t^\infty C_4 \cdot \|x(\tau)\|^2\, d\tau \leq \int_t^\infty C_4 \cdot K e^{-\lambda t} \|x_0\|^2\, d\tau$$

$$= \frac{K \cdot C_4}{\lambda} \|x_0\|^2$$

Thus: $P(t) \leq \sqrt{\frac{\kappa c_a}{\lambda}} \, I$

Further: $x(\tau) = x_0 + \int_t^\tau A(s)\, x(s)\, ds$

$$\|x(\tau)\| \geq \|x_0\| - \int_t^\tau L \|x(s)\|\, ds \quad \boxed{(\exists L \text{ s.t } \|A(s)\| \leq L)}$$

Define: $z(\tau) = \int_t^\tau L \|x(s)\|\, ds$, thus: $\frac{dz}{d\tau} = L\|x(\tau)\|$, but

$\|x(\tau)\| = \|x_0\| - z(\tau) + \Delta(\tau)$, for some nonnegative $\Delta(\tau)$, thus:

$$\frac{dz}{d\tau} = L\|x_0\| - Lz(\tau) + L\Delta(\tau)$$

$$z(\tau) = \int_t^\tau e^{-L(\tau-s)} \left(L\|x_0\| + \Delta(s) \right) ds$$

$$\leq \int_t^\tau e^{-L(\tau-s)} L\|x_0\|\, ds = \|x_0\| \cdot e^{-L(\tau-s)} \Big|_{s=t}^{s=\tau}$$

$$= \|x_0\| \cdot \left(1 - e^{-L(\tau-t)}\right)$$

Thus: $\|x(\tau)\| \geq \|x_0\| - z(\tau)$

$$\geq \|x_0\| - \|x_0\| \left(1 - e^{-L(\tau-t)}\right)$$

$$\geq \|x_0\| \cdot e^{-L(\tau-t)}$$

$$x_0^T P(t)\, x_0 = \int_t^\infty x^T(\tau)\, Q(\tau)\, x(\tau)\, d\tau \geq \int_t^\infty c_3 \cdot \|x(\tau)\|^2$$

$$\geq \int_t^\infty c_3 \cdot \|x_0\|^2 \cdot e^{-2L(\tau-t)}\, d\tau$$

$$= \frac{c_3}{2L} \|x_0\|^2$$

Thus: $P(t) \geq \sqrt{\frac{c_3}{2L}} \, I$, $P(t)$ is bounded and + def

Finally, we want to show that: $-\dot{P}(t) = Q(t) + A^T(t)\, P(t) + P(t)\, A(t)$

We need the following result:

$$\frac{\partial}{\partial t}\Phi(\tau,t) = -\Phi(\tau,t)A(t)$$

Proof: Use the fact that $\Phi(\tau,t) = \left(\Phi(t,\tau)\right)^{-1}$ and $\frac{d}{dt}(A(t))^{-1} = -A^{-1}(t)\frac{dA}{dt}A^{-1}(t)$

Therefore,

$$\frac{\partial}{\partial t}\Phi(\tau,t) = \frac{\partial}{\partial t}\left(\Phi(t,\tau)\right)^{-1} = -\left(\Phi(t,\tau)\right)^{-1}A(t)\,\Phi(\cancel{t},\tau)\cdot\left(\Phi(\cancel{t},\tau)\right)^{-1}$$

$$= -\left(\Phi(t,\tau)\right)^{-1}A(t)$$

$$= -\Phi(\tau,t)A(t)$$

Thus: $-\dot{P}(t) = \frac{d}{dt}\left[-\int_t^{\infty}\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau\right]$

$$= \Phi(t,t)Q(t)\Phi(t,t) + \int_t^{\infty}A^T(t)\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)\,d\tau$$

$$+ \int_t^{\infty}\Phi^T(\tau,t)Q(\tau)\Phi(\tau,t)A(t)\,d\tau$$

$$= Q(t) + A^T(t)P(t) + P(t)A(t)$$

Note: This theorem is one of the converse Lyapunov theorems:

Lyapunov: existence of Lyapunov function $\Longrightarrow$ stability

Converse: existence of Lyapunov function $\Longleftarrow$ stability

Theorem (4.13): Let $x=0$ be an equilibrium for

$$\dot{x} = f(x,t) \;;\; f: D \times [0,\infty) \Rightarrow \mathbb{R}^n \quad \text{.....} (*)$$

where f is continuously differentiable in D, and the Jacobian matrix $\frac{\partial f}{\partial x}(x,t)$ is bounded and Lipschitz in D, uniformly in t

Define $A(t) = \frac{\partial f}{\partial x}(x,t)\Big|_{x=0}$. If 0 is an exponentially stable equilibrium of

$\dot{x} = A(t)x$, then it is also an exponentially stable eq. for $(*)$

Example:

$$\dot{x}_1 = -x_1 + x_2$$

$$\dot{x}_2 = -\sin x_1 - g(t)x_2$$, where $g(t)$ is continuously differentiable

and $0 < a \le g(t) \le b < \infty$

$$\frac{\partial f}{\partial x}(x,t)\Big|_{x=0} = \begin{bmatrix} -1 & 1 \\ -\cos x_1 & -g(t) \end{bmatrix}\Bigg|_{x=0} = \begin{bmatrix} -1 & 1 \\ -1 & -g(t) \end{bmatrix}$$

Consider a system:

$$\dot{\hat{x}}_1 = -\hat{x}_1 + \hat{x}_2$$

$$\dot{\hat{x}}_2 = -\hat{x}_1 - g(t)\hat{x}_2$$

Define: $V(\hat{x},t) = \frac{1}{2}\hat{x}_1^2 + \frac{1}{2}\hat{x}_2^2$

$$\frac{dV}{dt} = -\hat{x}_1^2 - \hat{x}_1\hat{x}_2 - \hat{x}_1\hat{x}_2 - g(t)\hat{x}_2^2$$

$$= -\hat{x}_1^2 - g(t)\hat{x}_2^2$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & g(t) \end{bmatrix} \ge \min(1,a)$$

Thus: the origin is exponentially stable.