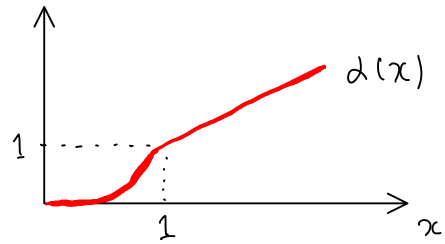
Comparison Functions

Def: A continuous function $\alpha : [0, a) \to [0, \infty)$ is a class K function if $\alpha(0) = 0$, and α is strictly increasing. If $a = \infty$ and $\alpha(x) \to \infty$ as $x \to \infty$ then α belongs to class K_∞

Example: $\alpha(x) = \min\{r, r^2\}$ is K_∞ (and thus also K)

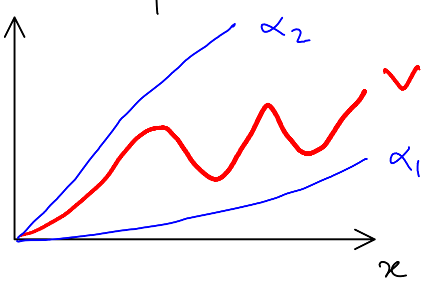


Lemma: Let $V : D \to R$ be a continuous + def function. D contains the origin. Let $B(0,r) \subset D$ for some $r > 0$. Then, there exist class K functions α_1 and α_2 defined or $[0, r]$ such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|)$$

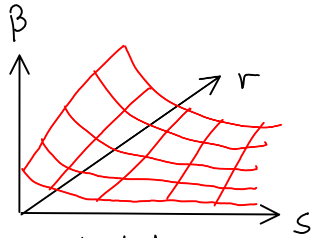
for all $x \in B(0,r)$.

If $D = \mathbb{R}^n$ and V is radially unbounded, then α_1 and α_2 can be chosen from K_∞ functions.



Interpretation: V can always be wedged between two class K functions.

Def: A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is a class KL function if for every s, $\beta(\cdot, s)$ is a class K function, and for every r, $\beta(r, \cdot)$ is decreasing and $\beta(r, s) \to 0$ as $s \to \infty$.



sketch

Lemma: Consider the scalar system:

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

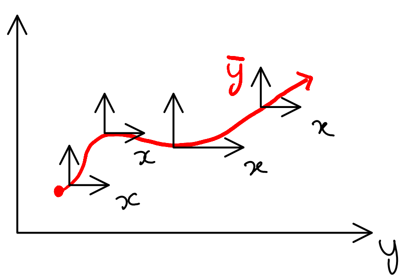
where α is a locally Lipschitz class K function defined on $[0, a)$. For all $0 \le y_0 < a$, the equation has a unique solution $y(t)$ defined on all $t \ge t_0$. Also, $y(t) = \sigma(y_0, t - t_0)$, with σ a class KL function defined on $[0, a) \times [0, \infty)$

Stability of Time Varying / Nonautonomous Systems

Systems of the form: $\dot{x} = f(x, t)$, $x \in \mathbb{R}^n$, $t \ge t_0$

Stability of nonautonomous systems can be applied towards stability around a solution trajectory.

Consider: $\dot{y} = g(y, \tau)$, and suppose that $\bar{y}(\tau)$ is a solution for $\tau \ge t_0$



Define $x(t) \triangleq y(t + t_0) - \bar{y}(t + t_0)$

$$\frac{dx}{dt} = g(y(t+t_0), t+t_0) - \dot{\bar{y}}(t+t_0)$$

$$= \underbrace{g(x(t) + \bar{y}(t+t_0), t+t_0) - \dot{\bar{y}}(t+t_0)}_{\triangleq f(x,t)}$$

Notice that $f(0, t) = 0$, $\forall t \ge 0$.

Stability can be non-uniform:

Example: $\dot{x} = (6t \sin t - 2t)x$ with initial condition $x(t_0)$

$$x(t) = x(t_0) \exp \int_{t_0}^{t} 6\tau \sin\tau - 2\tau \, d\tau$$

$$X(t) = X(t_0)\exp\Big[\underbrace{6\sin t - 6t\cos t - t^2 - 6\sin t_0 + 6t_0\cos t_0 + t_0^2}_{k(t,t_0)}\Big]$$

$\lim_{t\to\infty} k(t,t_0) = -\infty$, therefore $\sup_{t\geq t_0} k(t,t_0)$ exists. We denote it by $c(t_0)$

$X(t) \leq x(t_0)\, e^{c(t_0)}$. To show stability of the origin, for any given $\varepsilon > 0$, pick $\delta < \varepsilon \cdot e^{-c(t_0)}$ to get

$$|x(t_0)| < \delta \Rightarrow |x(t)| < \varepsilon, \forall t \geq t_0$$

However, we cannot pick δ independently of t_0. To see this, suppose $t_0 = 2\pi \cdot n$, for some integer n, and $t = 2\pi n + \pi$, then

$$x(t) = X(t_0)\exp\left[6(2\pi n + \pi) - (2\pi n + \pi)^2 + 6(2\pi n) + (2\pi n)^2\right]$$

$$= X(t_0)\exp\left[\underbrace{24\pi n + 6\pi - \pi^2 - 4\pi^2 n}_{(24\pi - 4\pi^2)n + 6\pi - \pi^2}\right]$$

Therefore: $\lim_{t\to\infty} \frac{x(t)}{x(t_0)} = \infty$

Def: An equilibrium $x = 0$ of a nonautonomous system is

- stable if for each $\varepsilon > 0$, there is a $\delta(\varepsilon, t_0) > 0$ such that

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon, \forall t \geq t_0$$

- Uniformly stable if δ can be independent of t_0, thus $\delta(\varepsilon)$ only.
- asymp. stable if it is stable and there exists a $c(t_0) > 0$ such that $x(t) \to 0$ as $t \to \infty$, for all $\|x(t_0)\| < c$
- uniformly asymp. stable if it is uniformly stable and $\exists c > 0$ such that $\|x(t_0)\| < c \Rightarrow \lim_{t\to\infty} x(t) = 0$ uniformly in t_0

$\forall \eta > 0$, $\exists T(\eta, c)$ such that $\forall t \geq (t_0 + T)$, $\|x(t)\| < \eta$

T cannot depend on t_0

- globally uniformly asymp. stable if
  * it is uniformly stable, and $\delta(\varepsilon)$ can be chosen such that $\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$
  * For any $c, \eta > 0$, there exists $T(c, \eta)$ such that

$$\|x(t_0)\| < c \implies \|x(t)\| < \eta \quad , \forall\, t \geq t_0 + T(c, \eta)$$