STOCHASTIC HYBRID SYSTEMS

Note Title 4/1/2006

Stochastic processes

A stochastic process X_t, $t \in T$ can be thought of as a family of random variables, indexed by the time set T.

Example: $T = \{1, 2, 3, \ldots\}$ and X_t is the outcome of throwing a die. Thus, $X_t \in \{1, 2, 3, 4, 5, 6\}$

$T = \{1, 2, \ldots, 365\}$ and X_t is the price of a company's share in the stock market.

$T = \mathbb{R}$ and X_t is the radio noise signal picked up by an antenna.

Suppose that $X_t \in \mathcal{X}$,

The expectation of a process X_t is given by

$$E X_t := \sum_{x \in \mathcal{X}} x \, p(X_t = x) \quad \text{or} \quad \int_{\mathcal{X}} x \, p(X_t \in [x, x+dx]) \, dx$$

The expectation of a process is not random

Conditional expectation can be defined by replacing the probabilities with conditional probabilities.

$$E(X_t \mid X_\tau = y) = \sum_{x \in \mathcal{X}} x \, p(X_t = x \mid X_\tau = y)$$

A process is called a martingale if :

$$E(X_t \mid X_s = y) = y, \quad \forall t \geq s$$

A process is called a super-martingale if

$$E(X_t \mid X_s = y) \leq y, \quad \forall t \geq s$$

A process is called a sub-martingale if

$$E(X_t \mid X_s = y) \geq y, \quad \forall t \geq s$$

Discrete time Markov chains [DTMC]

A discrete time Markov chain can be defined as a triple

$$S = (Q, R, R_0), \text{ where}$$

Q is the set of states

R is the probabilistic transition relation and

R_0 is the initial distribution of states.

$$P(x(0) = q) = R_0(q), \quad q \in Q$$

$$P(x(k+1) = q' \mid x(k) = q) = R(q', q)$$

When Q is finite, R can be expressed as a matrix R_{ij}.
Suppose that $Q = (q_1, q_2, \ldots, q_N)$
Denote $P(x(k) = q_i) =: p_i(k)$, $q_i \in Q$, then we have

$$P(x(k+1) = q_i) = P(x(k+1) = q_i \mid x(k) = q_1) \cdot P(x(k) = q_1) + \cdots + P(x(k+1) = q_i \mid x(k) = q_N) \cdot P(x(k) = q_N)$$

$$p_i(k+1) = \sum_{j=1}^{N} R_{ij} \, p_j(k)$$

$$\begin{pmatrix} p_1(k+1) \\ p_2(k+1) \\ \vdots \\ p_N(k+1) \end{pmatrix} = R \begin{pmatrix} p_1(k) \\ p_2(k) \\ \vdots \\ p_N(k) \end{pmatrix} \longrightarrow \boxed{p(k+1) = R \, p(k)}$$

Suppose that $f : Q \to \mathbb{R}$ is a function of the states, then $f(x(k))$ is a stochastic process.

The expectation of f:

$$E\Big(f(x(k))\Big) = \sum_{q \in Q} f(q)\, P(x(k) = q)$$

If $f(x(k))$ is a nonnegative supermartingale, then the following inequality holds.

$$\boxed{P\left(\exists x \geq 0 \text{ s.t. } f(x(k)) \geq z\right) \leq \frac{f(x(0))}{z}}$$

The inequality can be used for probabilistic safety verification a la barrier certificate.

Idea: Find an $f : Q \to \mathbb{R}_+$ such that

- $f(q_0) = 1$, q_0 is the initial state
- $f(q) \geq \Gamma > 1$, $\forall q \in \mathcal{X}_u$ (the unsafe set)
- $f(X_t)$ is a supermartingale, then by the inequality,

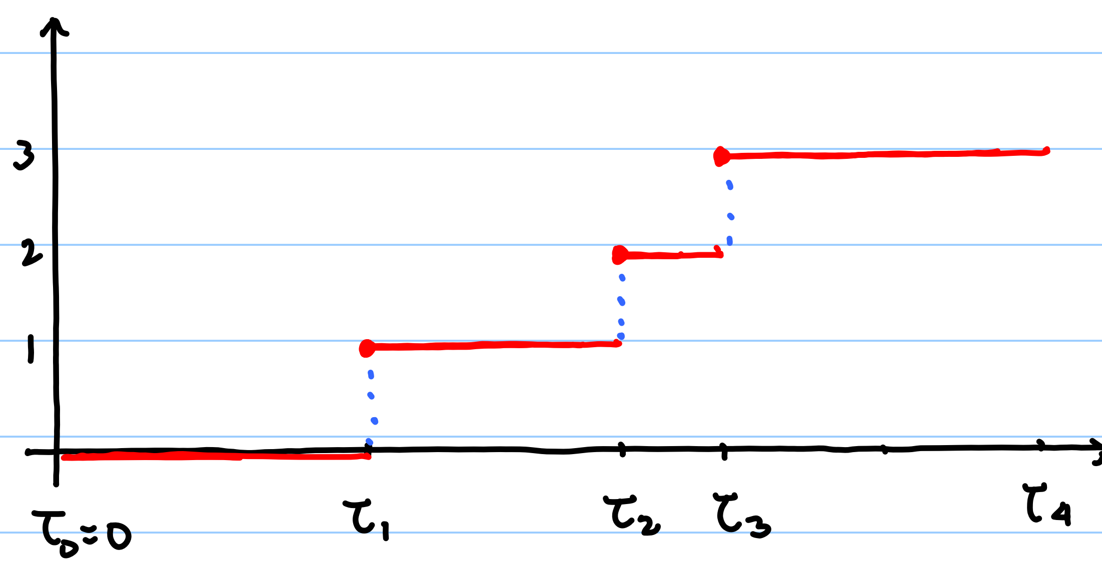
$$P(\exists k \geq 0, X_k \in \mathcal{X}_u) \leq \frac{1}{\Gamma}$$

Homework set 3

Poisson processes

A Poisson process P_t, $t \in \mathbb{R}_+$ with rate λ is given as:

$$P_0 = 0,$$
$$P_t = k, \quad \forall t \in [\tau_k, \tau_{k+1})$$



The time instants $\tau_1, \tau_2, \dots$ are random variables, and

- $P(\tau_k - \tau_{k-1} \geq x) = e^{-\lambda x}$
- $(\tau_k - \tau_{k-1})$ and $(\tau_i - \tau_{i-1})$ are independent for all $i \neq k$

$\tau_1, \tau_2, \dots$ are called arrival times. Poisson processes are typically used to model the occurrence of events, where only the average frequency of events is known.

Eg: The arrival of a phone call, the arrival of a customer at a shop, the malfunction of a system component.

Fixed interval approach

- Define a fixed time step $\Delta > 0$
- Set $P_0 = 0$
- Draw a random variable x_0 from the distribution

$$p(x = n) = \frac{(\lambda\delta)^n e^{-\lambda\delta}}{n!} \quad , \quad n \in \{0, 1, 2, 3, \ldots\}$$

- Set $P_\delta = x_0$
- Draw another random variable x_1 from the same distribution.
- Set $P_{2\delta} = x_0 + x_1$
- ... and so on

How to simulate a Poisson process with rate λ:

<u>Variable interval approach</u>

- Set $P_0 = 0$
- Draw a random variable δ_1 from the exponential distribution with rate λ
- Set $P_{\delta_1} = 1$, and $P_t = 0, \forall\, t \in [0, \delta_1)$
- Draw a random variable δ_2 from the exponential distribution with rate λ
- Set $P_{\delta_1+\delta_2} = 2$, and $P_t = 1, \forall\, t \in [\delta_1, \delta_1+\delta_2)$
- ... and so on.

Continuous Time Markov Chain [CTMC]

A CTMC is the continuous time version of DTMC.

A CTMC is characterized by a triple

$$S = (Q, R, R_0),$$

Q is the set of states, R the transition relation and R_0 is the distribution of initial states.

The transitions between states are triggered by Poisson processes $R : Q \times Q \to \mathbb{R}_+$, where $R(q, q')$ is the rate of the Poisson process corresponding to the transition from q' to q.

Define the rate matrix $L, \hat{L}$ as

$$L_{ij} \triangleq R(q_i, q_j)$$

$$\hat{L}_{ij} \triangleq \begin{cases} L_{ij} & \text{if } i \neq j \\ -\sum_i L_{ij} & \text{if } i = j \end{cases}$$

Define the probability vector $P_i(t)\big|_{i \in S}$ as

$P_i(t) \triangleq P(X_t = q_i)$, then we have that

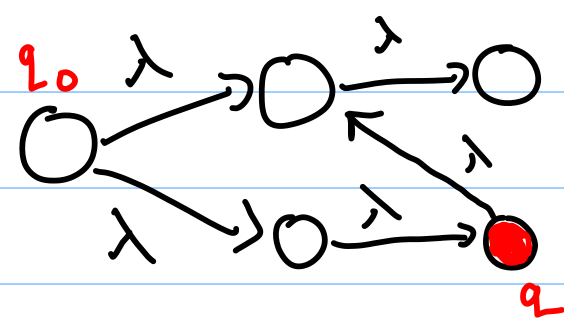
$$\boxed{\frac{dP}{dt} = \hat{L} P}$$

Similarly, any function $f : Q \to \mathbb{R}$ has the expectation

$$E f(x_t) = \sum_{q \in Q} f(q) P(X_t = q)$$

The probability of unsafety can be computed by removing all outgoing transitions from unsafe states and computing $P(X_t = q)$ for all $q \in X_u$.

E_q:



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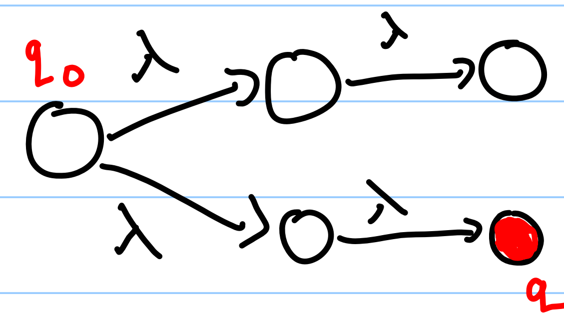


Intuition: the unsafe states absorb the probability density.

$P(\text{the state reaches the unsafe state in } [0,T]) = P(x_T = q \in \mathcal{X}_u)$

The same "trick" can be done to compute the unsafety probability for DTMC.

Brownian Motion

A standard brownian motion is a stochastic process $W_t \in \mathbb{R}$, $t \in \mathbb{R}_+$, where

- $W_0 = 0$
- $(W_t - W_s) \sim N(0, \sqrt{|t-s|})$

  Gaussian/normal distribution with mean 0 and variance $|t-s|$.
- $E(W_t W_s) = \min(t, s)$
- $(W_t - W_{t'})$ and $(W_s - W_{s'})$ are independent, $t \geq t' \geq s \geq s'$

The trajectories of W_t are continuous, but nowhere differentiable.

How to simulate a Brownian motion:

Continuous time simulation is not possible.

- Fix a time step $\delta > 0$.
- Set $W_0 = 0$
- Draw a random variable Δ_1 from the normal distribution $N(0, \sqrt{\delta})$
- Set $W_\delta = \Delta_1$
- Draw a random variable Δ_2 from the normal distribution $N(0, \sqrt{\delta})$
- Set $W_{2\delta} = \Delta_1 + \Delta_2$, and so on

Multimodal random dynamical system

A multimodal random dynamical system is given by

$$S = (Q, X, F, R, R_0), \text{ where}$$

Q is the set of discrete states,

$X = \mathbb{R}^n$ is the dimension of the continuous state space in each location.

$F(q) : \mathbb{R}^n \to \mathbb{R}^n$, the continuous state in location q evolves according to

$$\dot{x} = F(q, x)$$

R is the probabilistic transition relation between locations.

$R(q, q')$ is the rate of the Poisson process corresponding to the transition from q' to q.

R_0 gives the distribution of the initial states.

For any $q \in Q$ and $\Gamma \subset \mathbb{R}^n$,

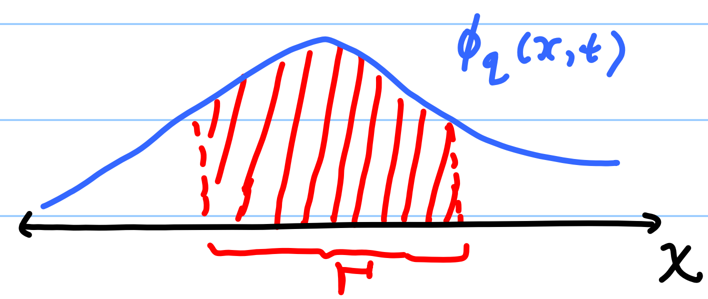
$$P(q(0) = q \text{ and } x(0) \in \Gamma) = \int_\Gamma R_0(q, x)\, dx$$

The execution:

The trajectory evolves following the continuous dynamics in each location. A transition can occur when a Poisson process generates an event. When this happens, the location changes accordingly, and hence the continuous dynamics switches.

Define the density function $\phi_q(x,t)$ as

$$P(q(t)=q \text{ and } x(t)\in\Gamma) = \int_\Gamma \phi_q(x,t)\,dx$$



$$\int_{\mathbb{R}^n} \phi_q(x,t)\,dx = P(q(t)=q), \text{ thus}$$

Total mass: $\displaystyle\sum_{q\in Q}\int_{\mathbb{R}^n} \phi_q(x,t)\,dx = 1, \ \forall t\geq 0$

The density functions $\phi_q(x,t)$ satisfy the PDE:

$$\frac{\partial \phi_q}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(F_i(q,x)\,\phi\right) + \sum_{q' \in Q} \lambda_{qq'}\, \phi_{q'}(x,t) - \sum_{q' \in Q} \lambda_{q'q}\, \phi_q(x,t)$$

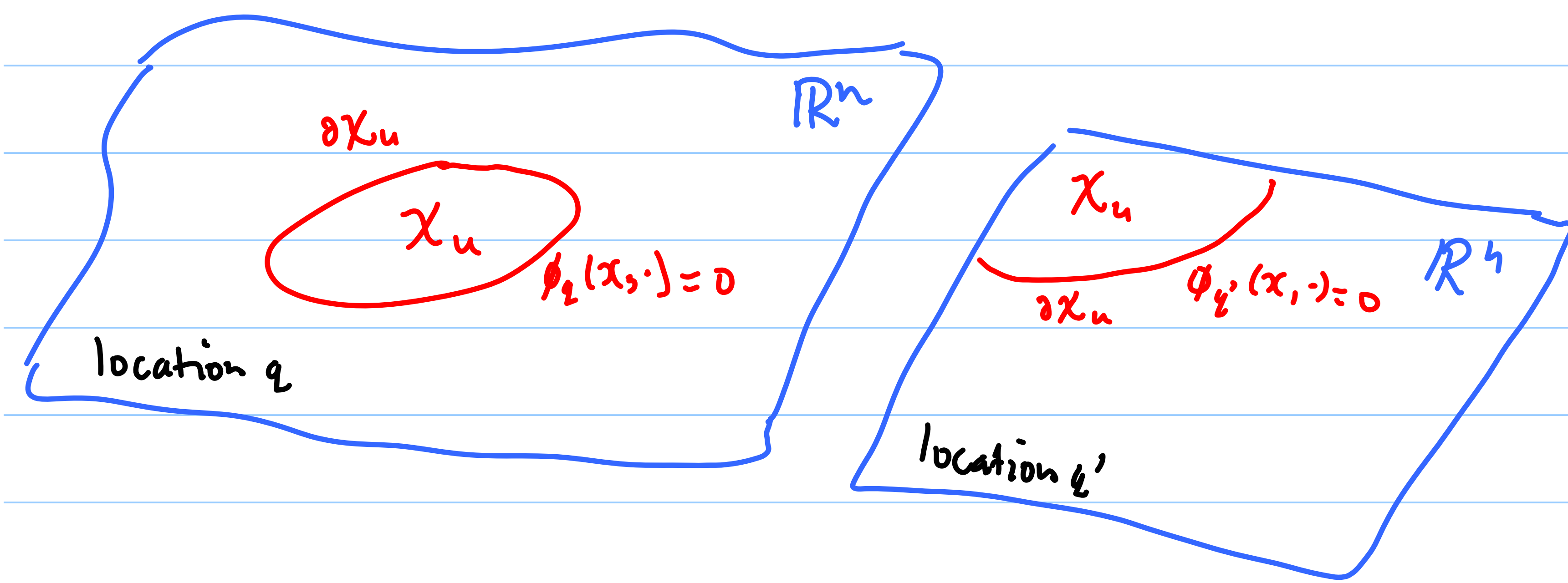
where $\lambda_{q'q}$ is the rate of the Poisson process corresponding to the transition from q to q'.

The first term is due to the continuous dynamics, the others are due to the switching.

The initial condition for the PDE:

$\phi_q(x,0) = R_0(q,x)$

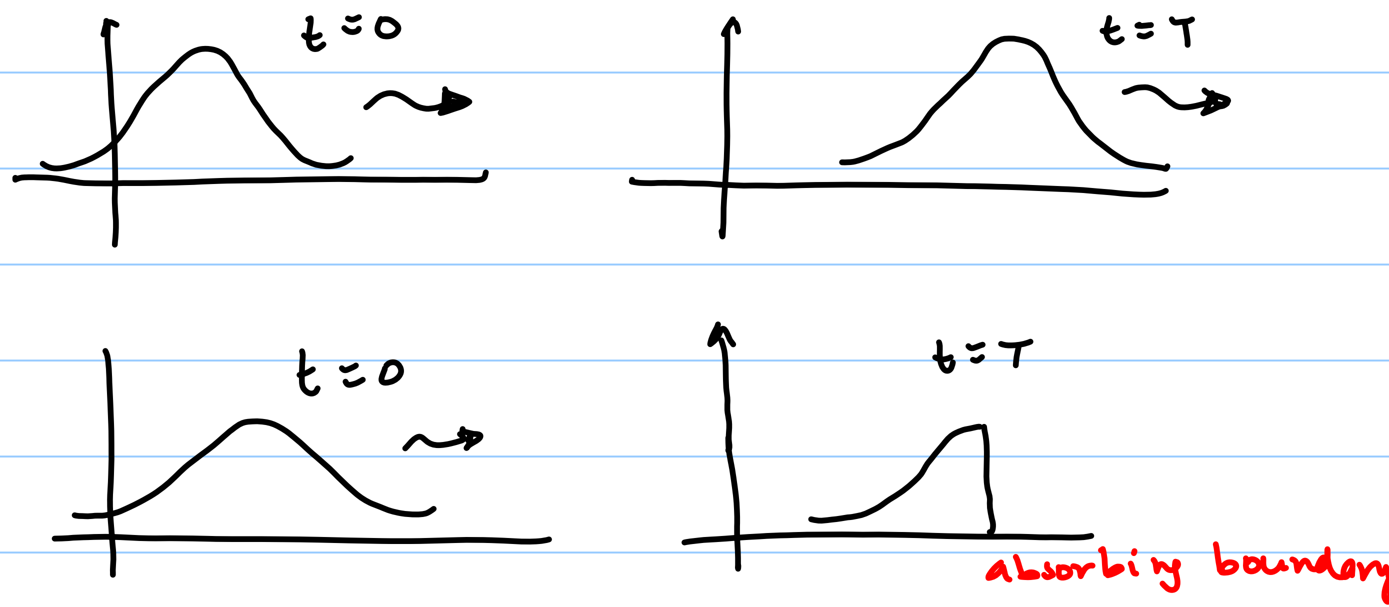
Unsafe set of states can be modeled as absorbing boundary condition.



Because of the absorbing boundary, the total mass is not conserved anymore.

$$\sum_{q \in Q} \int_{\mathbb{R}^n} \phi_q(x,t)\, dx =: \Omega(t) \leq 1$$

$1 - \Omega(T) = P$ (the state enters unsafe set in $[0,T]$)



Stochastic Differential Equation

A stochastic differential equation:

$$dx_t = a(x_t)\,dt + b(x_t)\,dW_t$$

$x_t \in \mathbb{R}^n$, with known initial distribution x_0

describes the evolution of a stochastic process x_t.

The equation is not written as:

$$\frac{dx_t}{dt} = a(x_t) + b(x_t)\frac{dW_t}{dt}$$

because W_t (standard Brownian motion) is not differentiable.

How to simulate realizations of a stochastic differential equation:

diffusion

$$dX_t = a(X_t)\,dt + b(X_t)\,dW_t$$

drift

- Draw a random variable x_0 from the initial distribution.
- Fixed a time step $\delta > 0$
- Draw a random variable $\Delta_1 \sim N(0, \sqrt{\delta})$
- Compute $x_1 = x_0 + a(x_0)\delta + b(x_0)\Delta_1$
- Draw a random variable $\Delta_2 \sim N(0, \sqrt{\delta})$
- Compute $x_2 = x_1 + a(x_1)\delta + b(x_1)\Delta_2$
- and so on ...

This approximation, as $\delta \to 0$ is called Ito integral.

The probability density function $\phi(x,t)$, where:

$$P(x_t \in \Gamma \subset \mathbb{R}^n) = \int_\Gamma \phi(x,t)\,dx$$

evolves following the PDE:

$$\frac{\partial \phi}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(a_i \phi) + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(b_i b_j \phi)$$

The first term is due to the drift of the SDE, the second term is due to the diffusion.