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Control of piecewise linear hybrid systems

A piecewise linear hybrid system consists of two parts:

- An automaton $(Q, E_{cd} \cup E_{in}, f)$, where Q is the set of states, E_{cd} are events that are triggered by the continuous dynamics (guard), E_{in} are input events (externally triggered), f represents the transition function

- For each $q \in Q$ we define an affine system

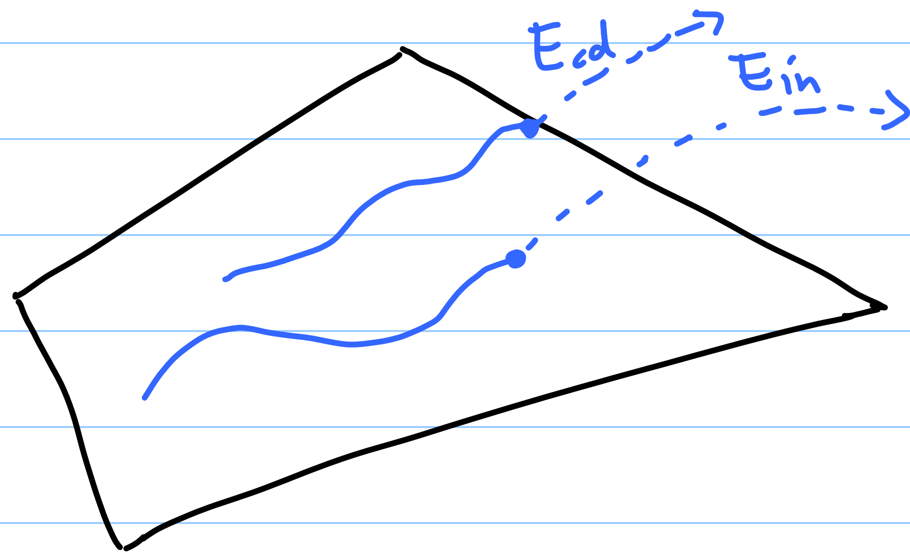
$$\dot{x}_q = A_q x_q(t) + B_q u(t) + a_q$$
$$y(t) = C_q X_q(t) + D_q U(t) + c_q$$

with $x_q \in \mathcal{X}_q$ and $u \in U$.

$\mathcal{X}_q$ and $\mathcal{U}$ are assumed to be polyhedral

When an event $e \in E_{in}^{-}$ or when the continuous state reaches the guard $G_q(e)$, a transition occurs and the state is reset according to:

$$q^+ = f(q^-, x_{q^-}, e)$$
$$x_{q^+}^+ = A_r(q^-, e, q^+)\, x_{q^-}^- + b_r(q^-, e, q^+)$$



Assumptions:

- At any fixed time only a finite number of discrete transitions can occur (no livelock).
- On any finite interval only a finite number of discrete transitions can occur (non zeno)

Reachability:

A state $(q,x) \in Q \times X$ is reachable from the initial state $(q_0, x_{q_0,0}) \in Q \times X$ if there exist two sequences

- (t_i, e_i), $i = 1, \dots, m$, $e_i \in E_{in}$, $t_i < t_j$, $j > i$
- $u_i : [t_i, t_{i+1}) \to U$, $i = 1, 2, \dots, m$

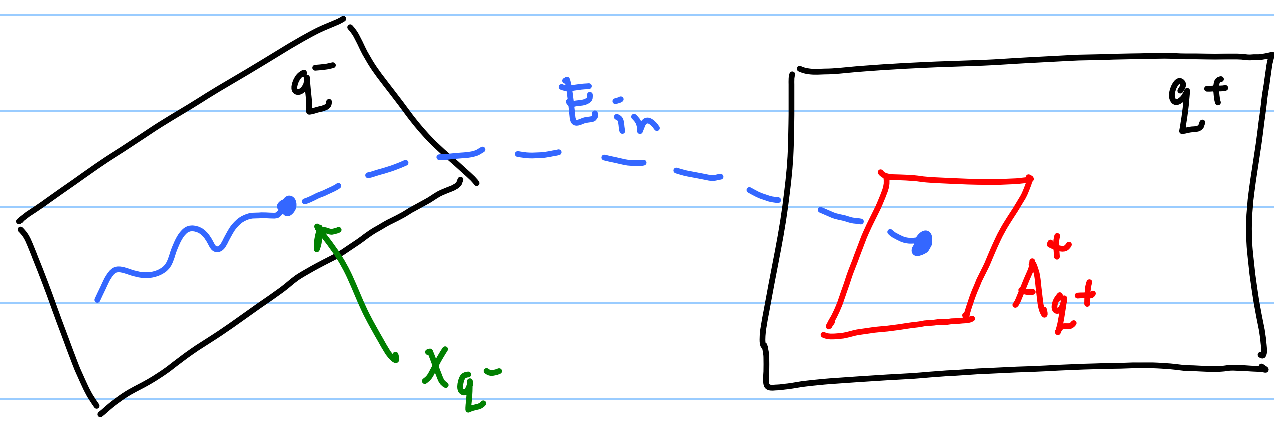
Such that by applying the input events e_i and input signals u_i, we steer the state from $(q_0, x_{q_0,0})$ to (q, x) in finite time

If every state is reachable from the initial state, then the system is said to be reachable.

A departure set (or exit set) of the system is defined to be :

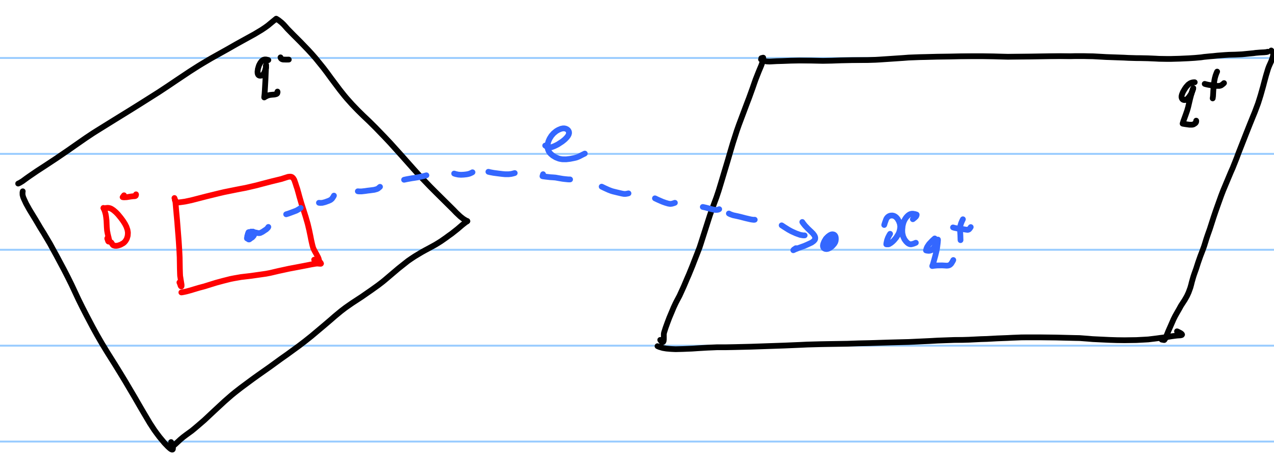
- A guard $G_q(e)$, $\forall e \in E_{cd}$, $q \in Q$ or
- $D(q^-, e, q^+, A^+_{q^+}) = \{ x_{q^-} \in \mathcal{X}_{q^-} \mid q^+ = f(q^-, x_{q^-}, e)$ and

$$A_r(q^-, e, q^+) x_{q^-} + b_r(q^-, e, q^+) \in A^+_{q^+} \}$$

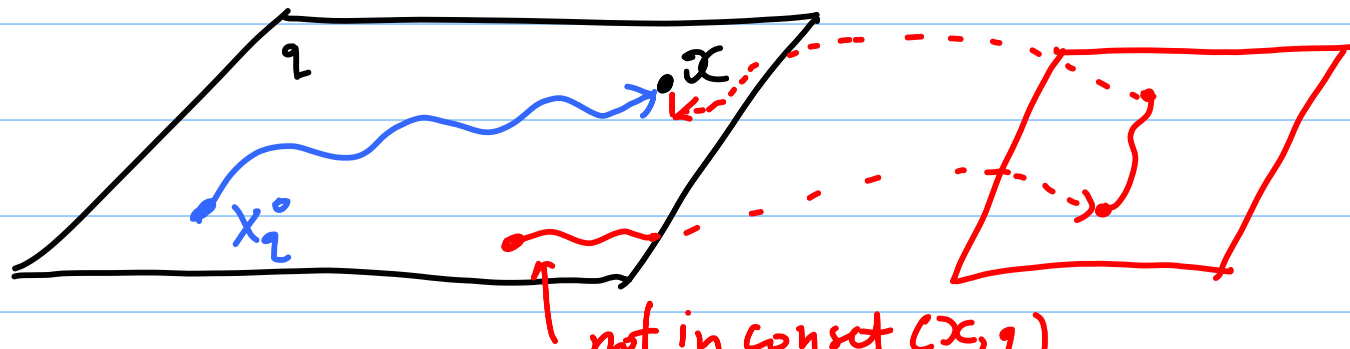


An arrival set (or entry set) is defined to be a set of the form

$$AR(q^-, e, q^+, D^-) = \{ x_{q^+} \in X_{q^+} \mid \exists\, x_{q^-} \in D^- \subseteq X_{q^-} \text{ s.t. } q^+ = f(q^-, x_{q^-}, e),\ x_{q^+} = A_r(q^-, e, q^+) x_{q^-} + b_r(q^-, e, q^+) \}$$



The controllability set (conset) of $(q, x) \in Q \times \mathcal{X}$ is the set of all states $x_q^\circ \in \mathcal{X}_q$ from which x can be reach without leaving $\mathcal{X}_q$.

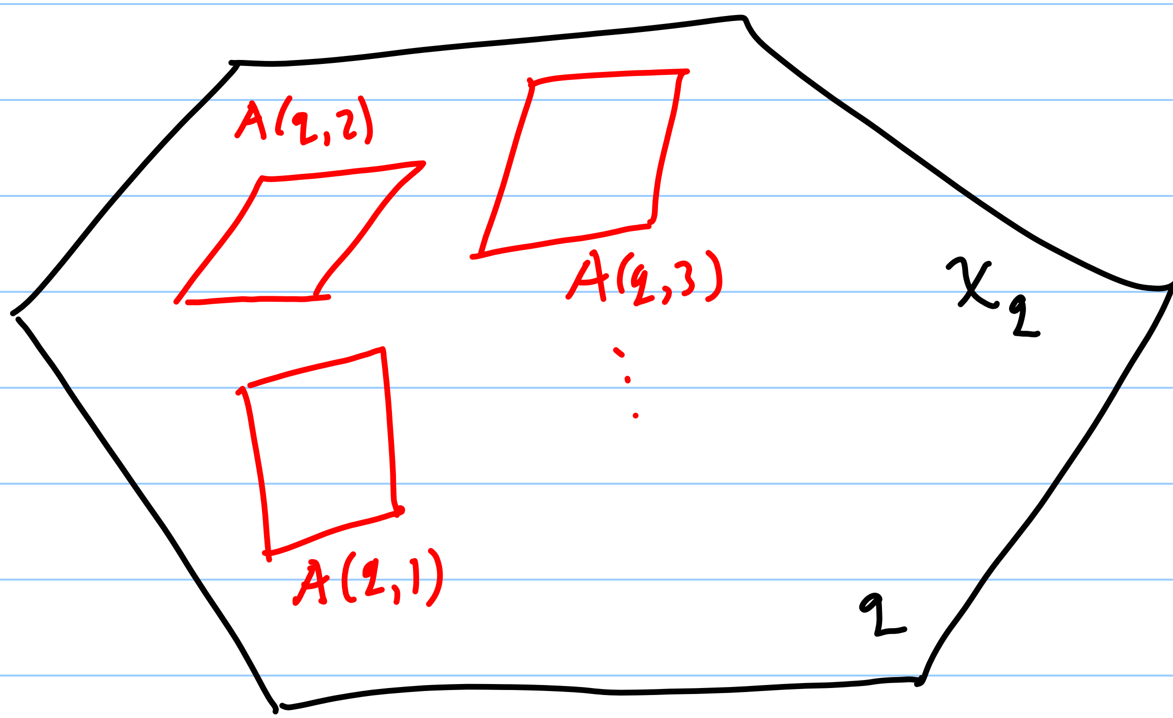


For any set $S_q \subseteq \mathcal{X}_q$,

$$\mathrm{conset}(q, S_q) := \bigcup_{x_q \in S_q} \mathrm{conset}(q, x_q)$$

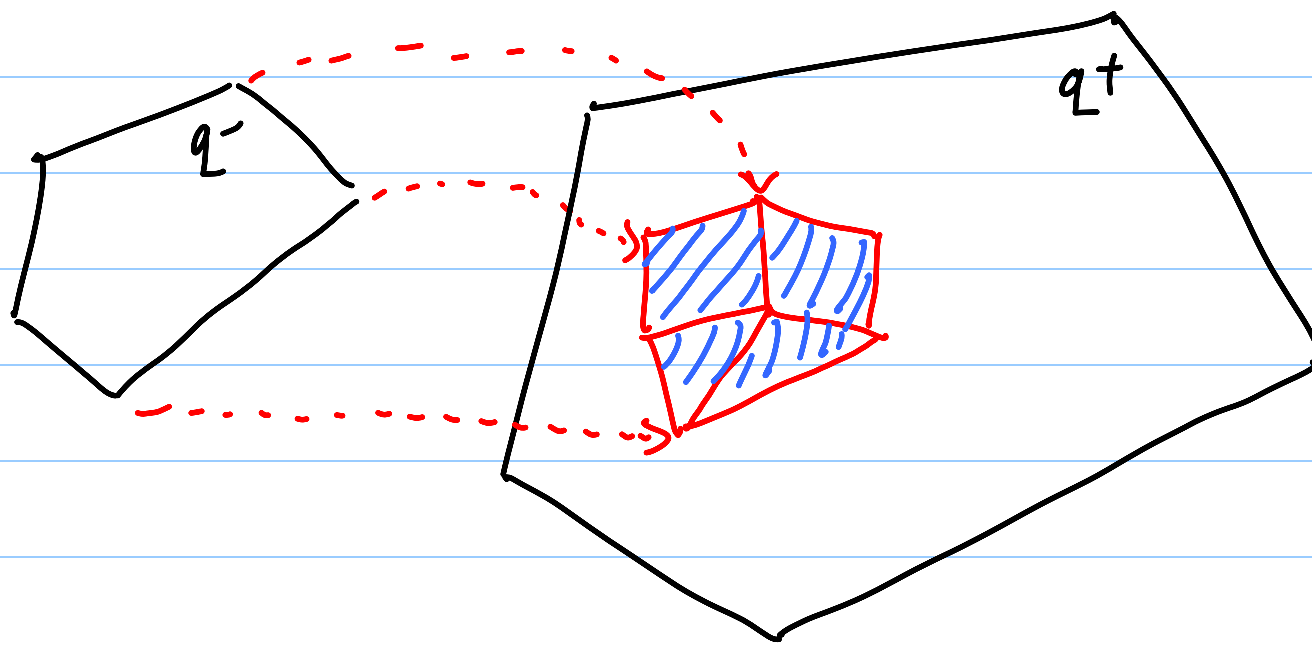
Assume that there exists a collection of disjoint sets of the form

$$A = \{ A(q,k) \subseteq X_q \mid k \in \{1,2,\dots,n_q\}, q \in Q \}$$



Assume that for every arrival set $AR(q^-, e, q^+, \mathcal{X}_{q^-})$, $q^-, q^+ \in Q$, $e \in E_{in} \cup E_{ed}$

$$AR(q^-, e, q^+, \mathcal{X}_{q^-}) \subseteq \bigcup_{k \in \{1,2,\dots,n_{q^+}\}} A(q^+, k)$$

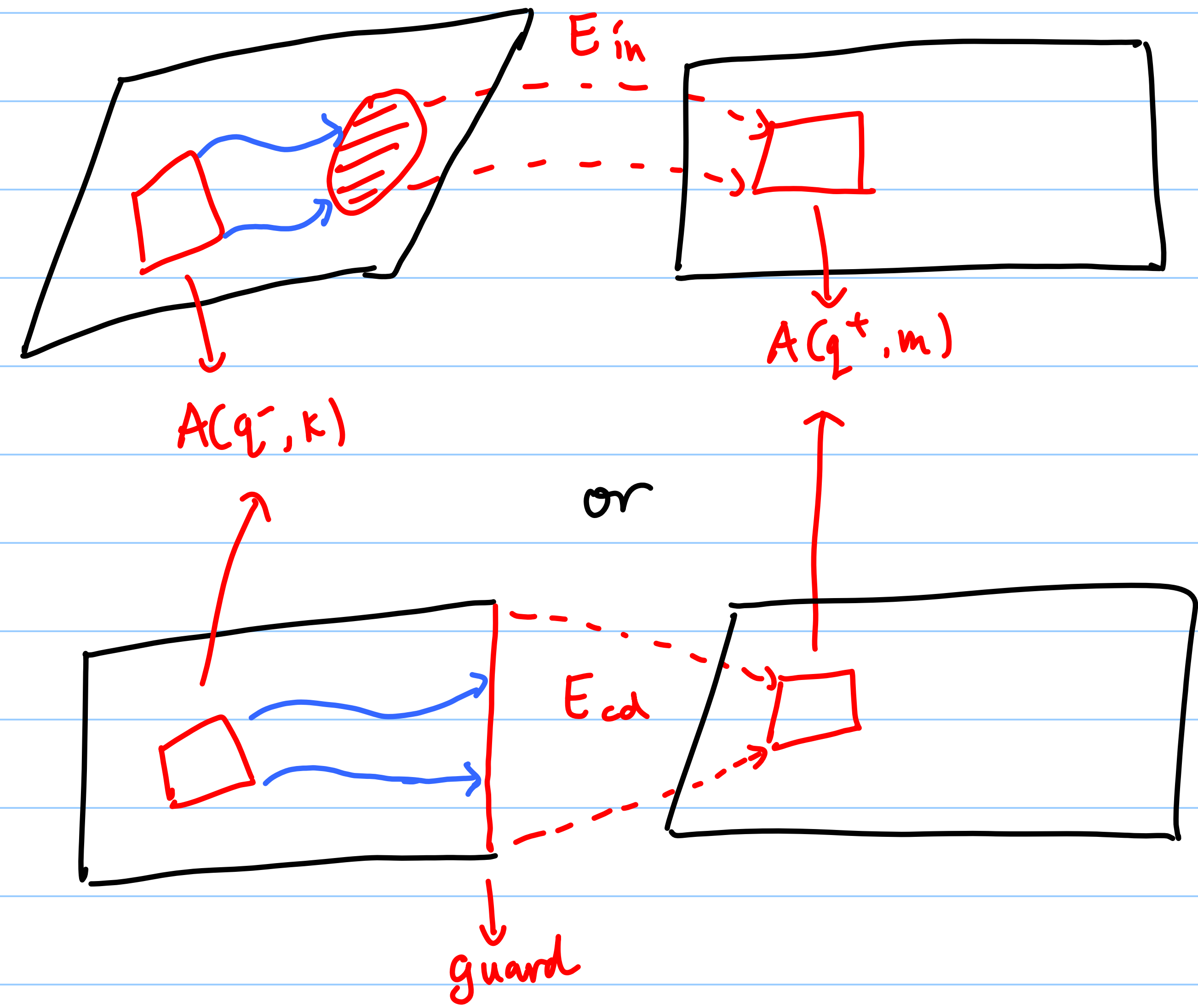


We can build an automaton $(A, E_{in} \cup E_{od}, f_A, A_0)$,

where the transition function f_A is defined as:

$$f_A(A(q^-,k),e) = \begin{cases} A(q^+,m) \text{ if } A(q^-,k) \subseteq \mathrm{conset}(D(q^-,e,q^+,A(q^+,m))) \\ \text{or} \\ A(q^-,k) \subseteq \mathrm{conset}(G_{q^-}(e)) \text{ and} \\ A_r(q^-,e,q^+)x_{q^-} + b_r(q^-,e,q^+) \in A(q^+,m), \text{ for} \\ \text{all } x_{q^-} \in G_{q^-}(e) \end{cases}$$

The automaton is possibly non-deterministic.



E_{in}
$A(q^-_i, k)$
$A(q^+_i, m)$
or
E_{cd}
guard

Reachability result: The piecewise linear hybrid systems is reachable if and only if:

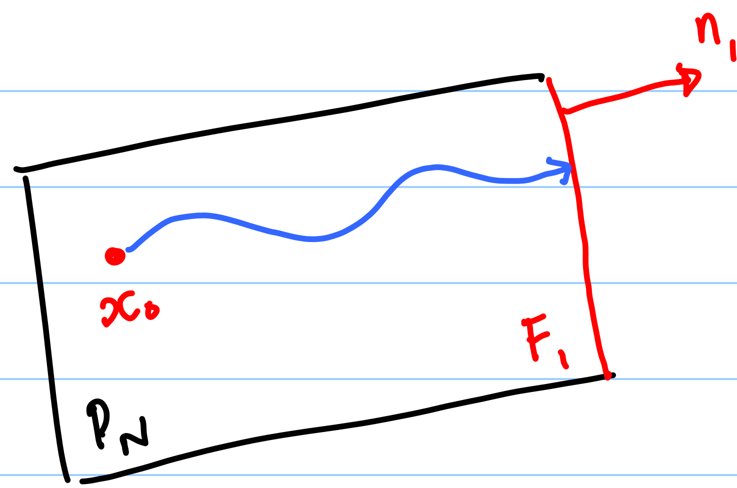
- The automaton $(A, E_{in} \cup E_{cd}, f_A, A_0)$ is reachable, and
- For any $(q, \tilde{x}) \in Q \times \mathcal{X}$, there exists an $A(q,k)$ such that

$$A(q,k) \subseteq \text{conset}(q, \tilde{x})$$

Interpretation: The first condition takes care reaching the sets in A, the second condition takes care reaching any set from A.

Construction of the finite set A can be done with an iteration similar to the bisimulation algorithm.

Problem:

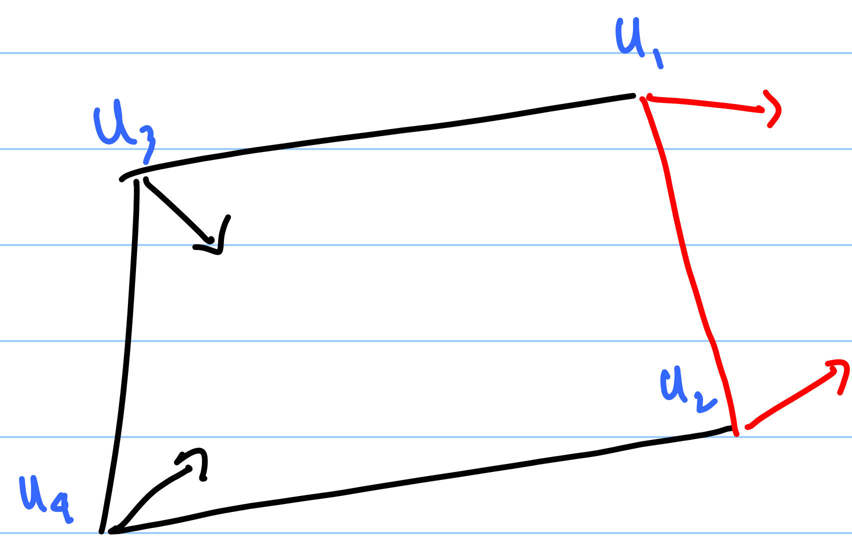


How to design the control input, such that the state exits the polytope P_N through the facet F_1?

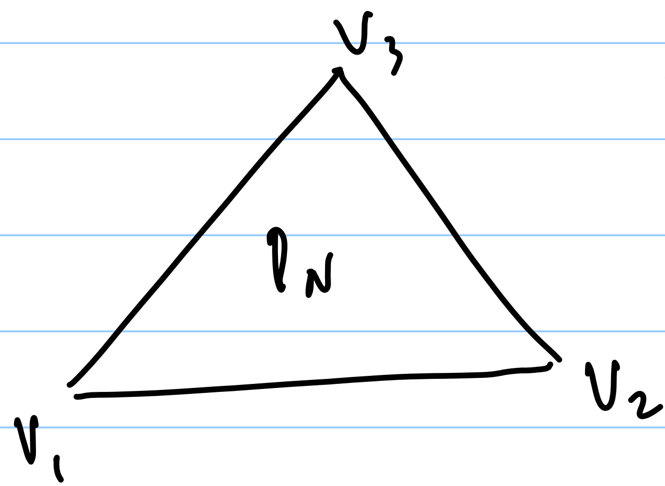
Find $T \geq 0$, and $u : [0, T] \to U$ such that:

- $x(t) \in P_N$, $\forall t \in [0, T]$
- $x(t) \in F_1 \iff t = T$
- $n_1 \cdot \dot{x}(T) > 0$ (the state is going out)

Necessary condition: Full dimensional polytopes. Check at the vertices of the polytope. There must be input values such that the velocity vector is pointing outward at the exit facet, and inward at other facets.



If P_N is a simplex, any point in P_N can be written as a unique convex combination of the vertices



$$\forall x \in P_N, \exists \lambda_1, \lambda_2, \lambda_3 \in [0,1] \text{ s.t}$$

$$x = \sum_{i=1}^{3} \lambda_i V_i \; ; \quad \lambda_1 + \lambda_2 + \lambda_3 = 1$$

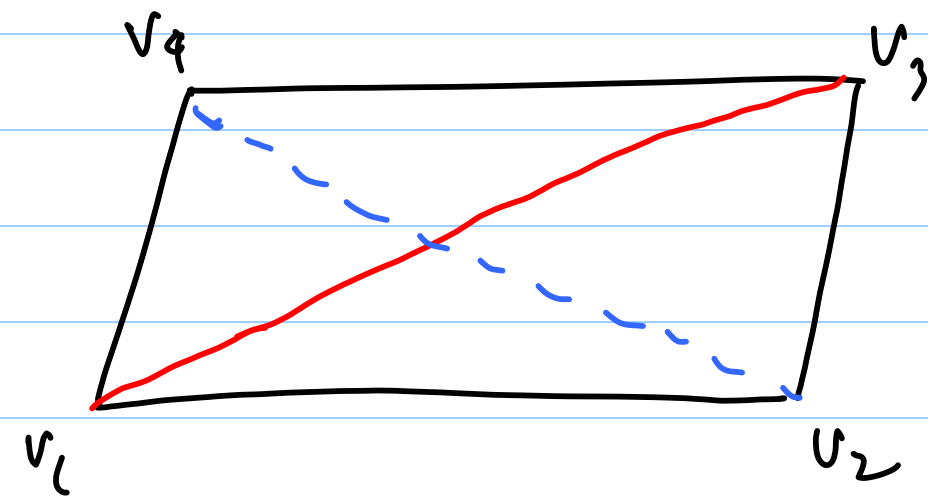
The control input is then defined as:

$$u = f(x) = \sum \lambda_i u_i$$

, where u_i are the control inputs that satisfy the necessary conditions.

For general polytopes (not necessarily simplices), the approach can be used by breaking down the polytope into simplices

E.g.: in $\mathbb{R}^2$, triangulation.



the apply the method for the simplices,

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Optimal control of linear discrete time hybrid systems

$$x(t+1) = A_i x(t) + B_i u(t) + f_i \;, \text{if}$$

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \;; \quad \mathcal{X}_i := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \,\middle|\, H_i x + J_i u \le K_i \right\}$$

$\mathcal{X}_i$, $i \in \{0, 1, \dots, s-1\}$ are polyhedral partition of the state+input space.

$x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_\ell}$ $\longrightarrow$ continuous + discrete states

$u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_\ell}$ $\longrightarrow$ continuous + discrete inputs

Model Predictive Control:

MPC is an optimization based controller design

Consider the optimization problem based on the following cost

$$J(u,\tilde{x}) = \sum_{k=0}^{\infty} L(x(k),u(k)) \quad , \quad x(0) = \tilde{x}$$

for the system

$$x(t+1) = A_i\, x(t) + B_i\, u(t) + f_i \quad , \text{if} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i$$

Because of the infinite time horizon, the solution is difficult to obtain.

One approximation is by using finite horizon

$$\hat{J}(u, \tilde{x}) = \sum_{k=0}^{T-1} L(x(k), u(k)) \quad + \tilde{L}(x(T)),$$

$$x(0) = \tilde{x}.$$

Informal reasoning: if the infinite sum converges, the the tail must be small. Hence, taking T Large enough, we can get a good approximation.

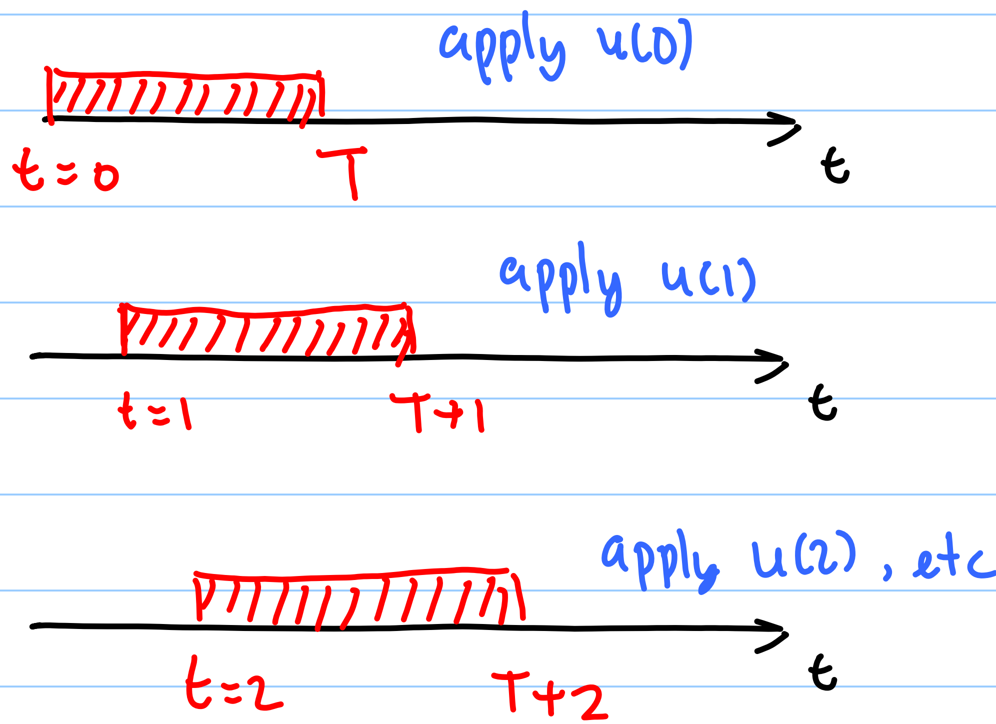
Finite horizon implies finitely many variables, on which the optimization must be performed, i.e. $u(0), u(1), \dots, u(T-1)$
The problem is more tractable.

The idea of finite horizon is combined with receding horizon.

Algorithm:

- Initialize the problem at $t=0$
- Compute the optimal $u(0), u(1), \dots, u(T-1)$
- Apply $u(0)$
- Restart the problem with the new state $(x(1))$

- Compute the optimal $u(1), \ldots, u(T)$
- Apply $u(1)$, and so on.



Finite time constrained optimal control:

Constraints:

$$\underbrace{\begin{aligned} u_{min} &\le u(t) \le u_{max} \\ x_{min} &\le x(t) \le x_{max} \end{aligned}}_{\text{defines a polyhedron } D}$$

$$x(t+1) = A_i x(t) + B_i u(t) + f_i \quad \text{if}$$

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i, \qquad \tilde{\mathcal{X}}_i := \mathcal{X}_i \cap D$$

Define the cost function:

$$J(u_0^{T-1}, x(0)) := \sum_{k=0}^{T-1} \|Q(x(k)-x_e)\|_p + \|R(u(k)-u_e)\|_p + \|P(x(T)-x_e)\|_p$$

Interpretation: p is user defined / problem specific norm definition.

x_e is the desired equilibrium state.

u_e is the desired input function.

If $p=2$, $\|Qx\|_2 = (x^T Q^T Q x)^{1/2} \sim$ quadratic cost

$$J(U_0^{T-1}, x(0)) := \sum_{t=0}^{T-1} (x(t)-x_e)^T Q (x(t)-x_e) + (u(t)-u_e)^T R (u(t)-u_e)$$
$$+ (x(T)-x_e)^T P (x(T)-x_e)$$

with $Q, P \geq 0$, $R > 0$

The solution of the optimal control problem takes the form of piecewise affine state feedback control law.

$$u(t) = F(i,t)\, x(t) + G(i,t) \quad \text{if } x(t) \in \mathcal{P}(i,t)$$

where $P(i,t)$ is a partition of the state space, given by

$$P(i,t) := \{ x \mid x^T L(i,t) x + M(i,t) x \leq N(i,t) \}$$

How the solution is obtained:

- Transform the dynamics of the system into mixed logic dynamics (MLD)

$$x(t+1) = \Phi x(t) + G_1 v(t) + G_2 \delta(t) + G_3 z(t)$$

$$E_2 \delta(t) + E_3 z(t) \leq E_1 v(t) + E_4 x(t) + E_5$$

together with the quadratic cost function, the problem becomes a Mixed Integer Quadratic Programming.

Tool: MPT toolbox for MATLAB