STABILITY OF HYBRID SYSTEMS

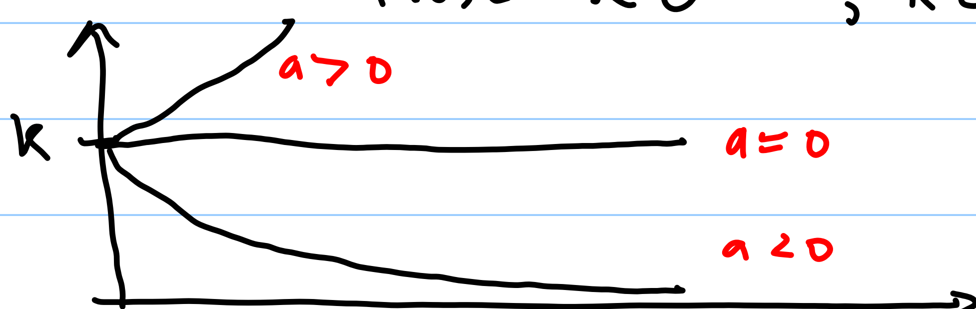
Note Title 3/12/2006

Review on stability

Consider scalar linear system: $\dot{x} = ax$, $x \in \mathbb{R}$

Trajectories of the system assume the form

$$x(t) = k e^{at}, \quad k \in \mathbb{R} \text{ constant}$$



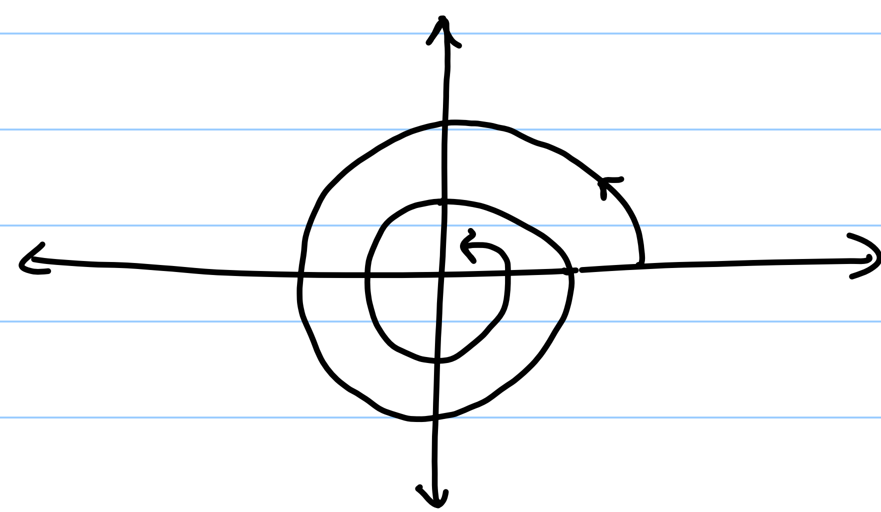
For multivariable systems: $\dot{x} = Ax$, $x \in \mathbb{R}^n$,
the system is stable (all trajectories converge to 0) iff the eigenvalues of A have negative real part.

Eigenvalues: the roots of $\det(sI - A)$

Example: $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, the eigenvalues are -1 and -2
thus stable

Example: $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, $(sI - A) = \begin{pmatrix} s+1 & 1 \\ -1 & s+1 \end{pmatrix}$,

$(s+1)^2 + 1 = 0 \implies s = -1 \pm i$, thus stable

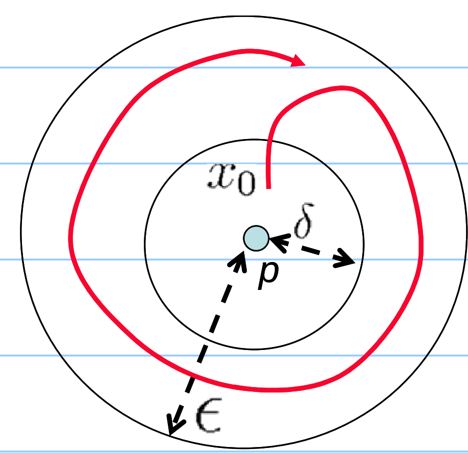


Nonlinear systems: Let the dynamics of the system be given by

$$\boxed{\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n}$$

The origin is an equilibrium. The origin is stable if for every $\varepsilon > 0$, there exists an $\delta > 0$ such that

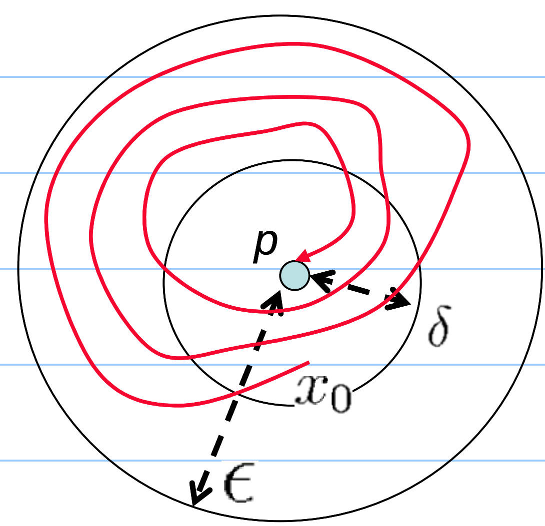
$$\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon, \quad \forall t \in \mathbb{R}_+$$



The origin is asymptotically stable if for every $\varepsilon > 0$, there exists an $\delta > 0$ such that

$$\|x(0)\| < \delta \implies \|x(t)\| < \varepsilon, \ \forall t \in \mathbb{R}_+, \text{ and}$$

$$\lim_{t \to \infty} x(t) = 0$$



Lyapunov function.

Stability can be verified by constructing a Lyapunov function.

Let a smooth function $V : \mathbb{R}^n \to \mathbb{R}$ be such that:

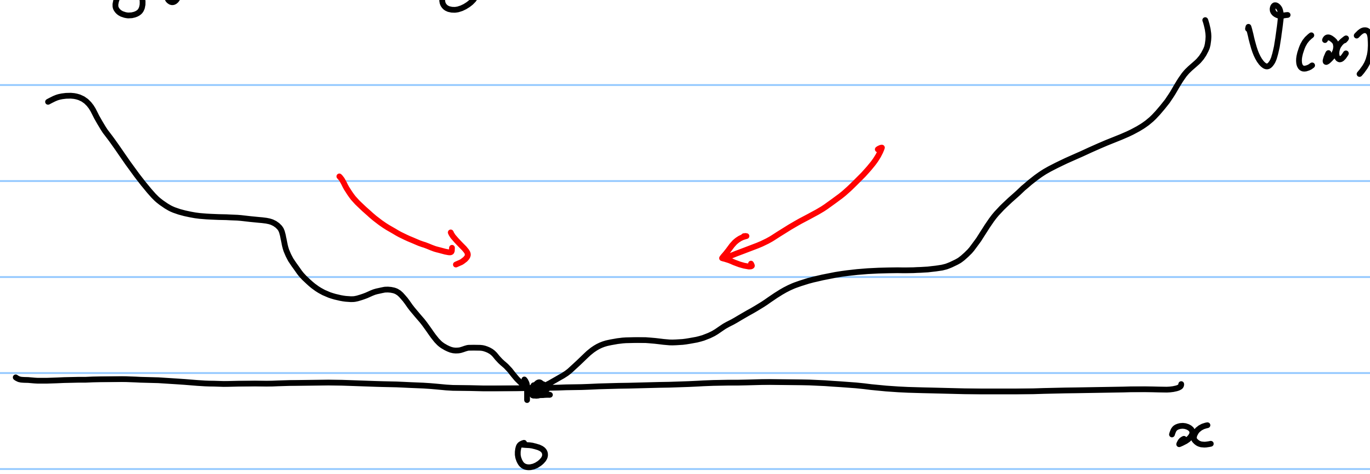
- $V(0) = 0$
- $V(x) > 0, \quad x \neq 0$

If : $\frac{dv}{dt} = \frac{\partial v}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial v}{\partial x} \cdot f(x) \leq 0, \forall x \in \mathbb{R}^n$

then the origin is stable.

If : $\frac{\partial v}{\partial x} \cdot f(x) < 0, \forall x \in \mathbb{R}^n$, then the origin is asymptotically stable

Interpretation: The Lyapunov function can be thought of as the energy in the system.



stable $\rightarrow$ it's possible to orbit on equipotential curve.

asymp. stable $\rightarrow$ must converge to zero energy level (the origin)

Challenge: how to compute / construct the Lyapunov function? (recall barrier certificate)

For linear systems: $\dot{x} = Ax$, we assume that the Lyapunov function is a quadratic function.

$$V(x) = x^T P x \text{, where } P \text{ is positive definite}$$

$$\frac{\partial v}{\partial x} = (x^T P + x^T P^T) \longrightarrow \frac{dv}{dt} = x^T (PA + P^T A) x$$

Goal: Find P such that $\boxed{PA + P^T A = Q < 0}$

Lyapunov equation

$$PA + P^T A = Q < 0 \qquad (*)$$

Thm: If A is stable then for any $Q < 0$, there exists a symmetric positive definite P such that $(*)$ is satisfied.

Thus, we can always find a quadratic Lyapunov function.

For nonlinear systems, there is no generic method, but, assuming polynomial vector fields, a polynomial Lyapunov function can be searched using semidefinite programming. E.g. SOSTOOLS (remember barrier certificate)

Hybrid systems

Switched linear systems:

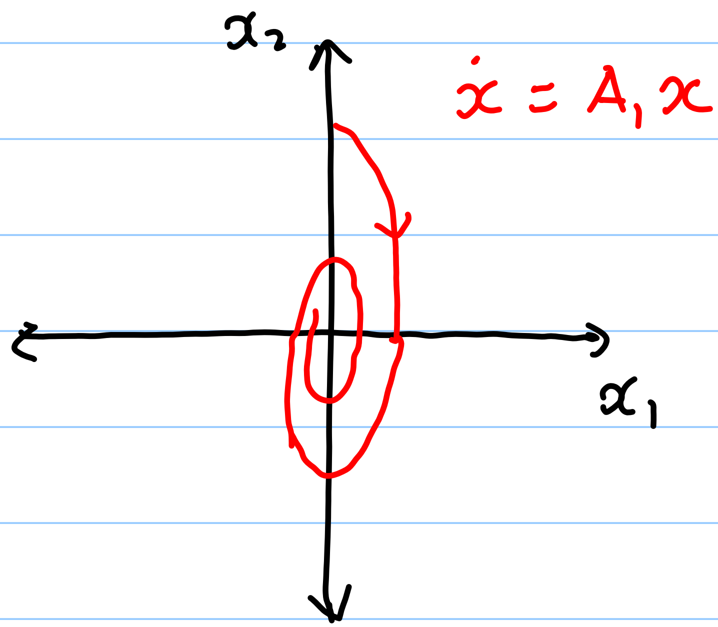
$$\dot{x} = A_{p(t)}\, x, \qquad p(t) \in \{1, 2, \dots m\}$$
$$x \in \mathbb{R}^n$$

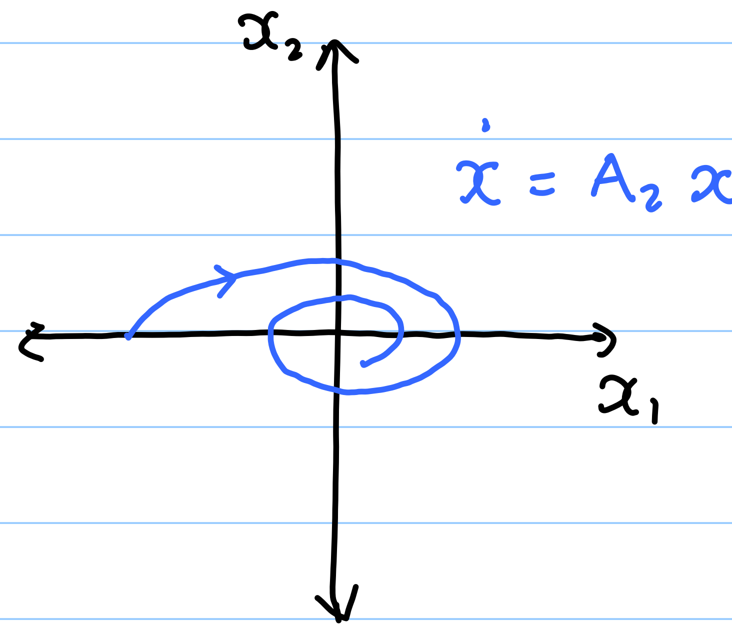
$p(t)$ indicates mode selection.

Notice that the state trajectories are continuous

The stability of the switched system cannot be inferred from the stability of the individual modes.

Consider the following case, where $m=2$, $n=2$

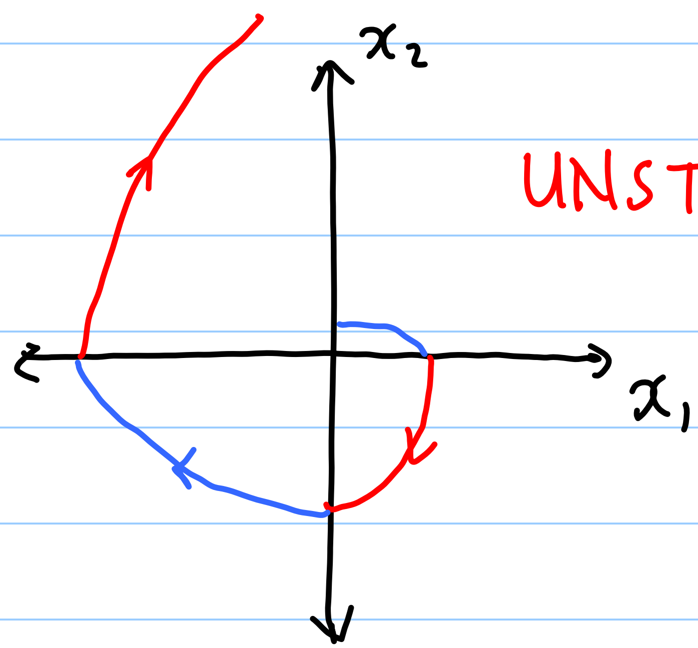




Observe that both modes are stable. However switching can results in instability.

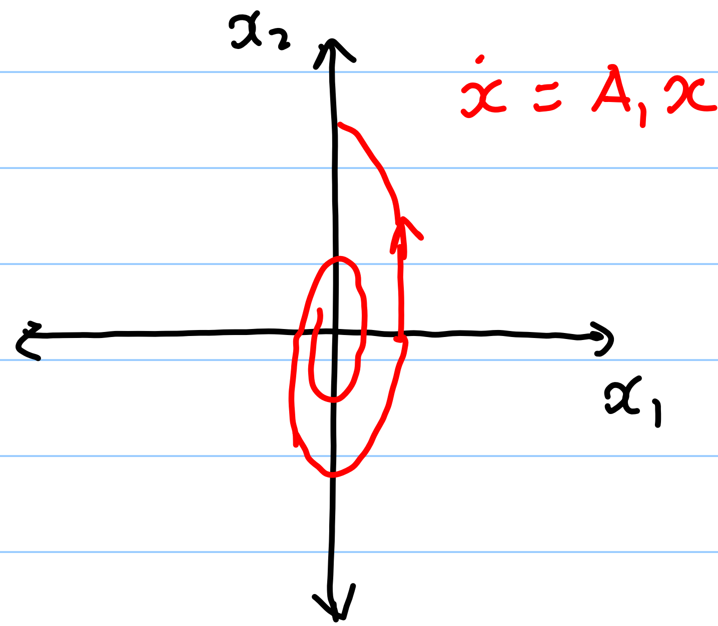
Consider : $\dot{x} = A_{p(t)}\, x$, $p \in \{1, 2\}$, where

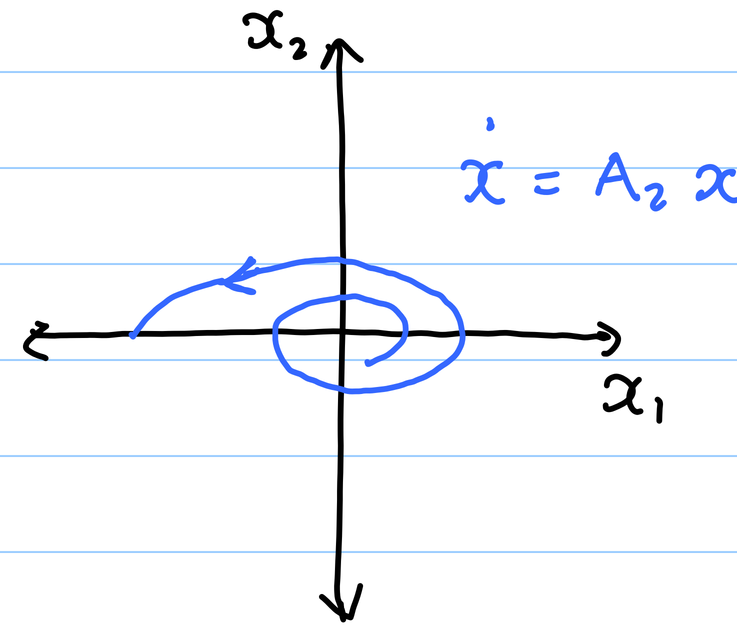
$$p(t) = \begin{cases} 1, & x_1 x_2 < 0 \\ 2, & x_1 x_2 \geq 0 \end{cases}$$



UNSTABLE!!

On the other hand, consider:

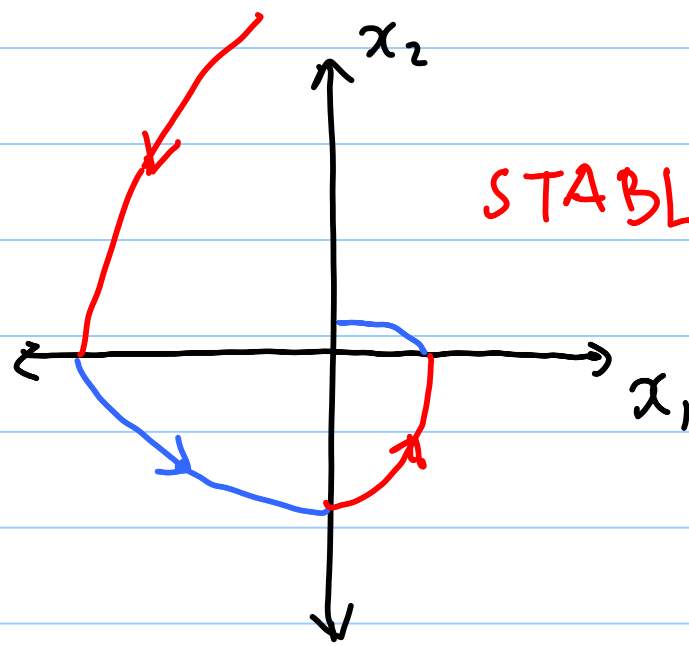




Both modes are unstable. Switching can result in stabili-ty.

Consider : $\dot{x} = A_{p(t)}x$, $p \in \{1, 2\}$, where

$$p(t) = \begin{cases} 1, & x_1 x_2 < 0 \\ 2, & x_1 x_2 \geq 0 \end{cases}$$



STABLE !!

Given a family of stable systems:

$$\dot{x} = f_{p(t)}(x) \quad . \quad p(t) \in \{1, 2, \dots, m\}$$

Suppose that all the modes are stable.

Question: What is the condition, under which any switching results in stable system?

Answer: If there exists a common Lyapunov function for all the modes, then any switching is stable.

Existence of a common Lyapunov function is rare.

Suppose that $V_i(x)$, $i \in \{1, 2, \dots, m\}$ is a Lyapunov function for

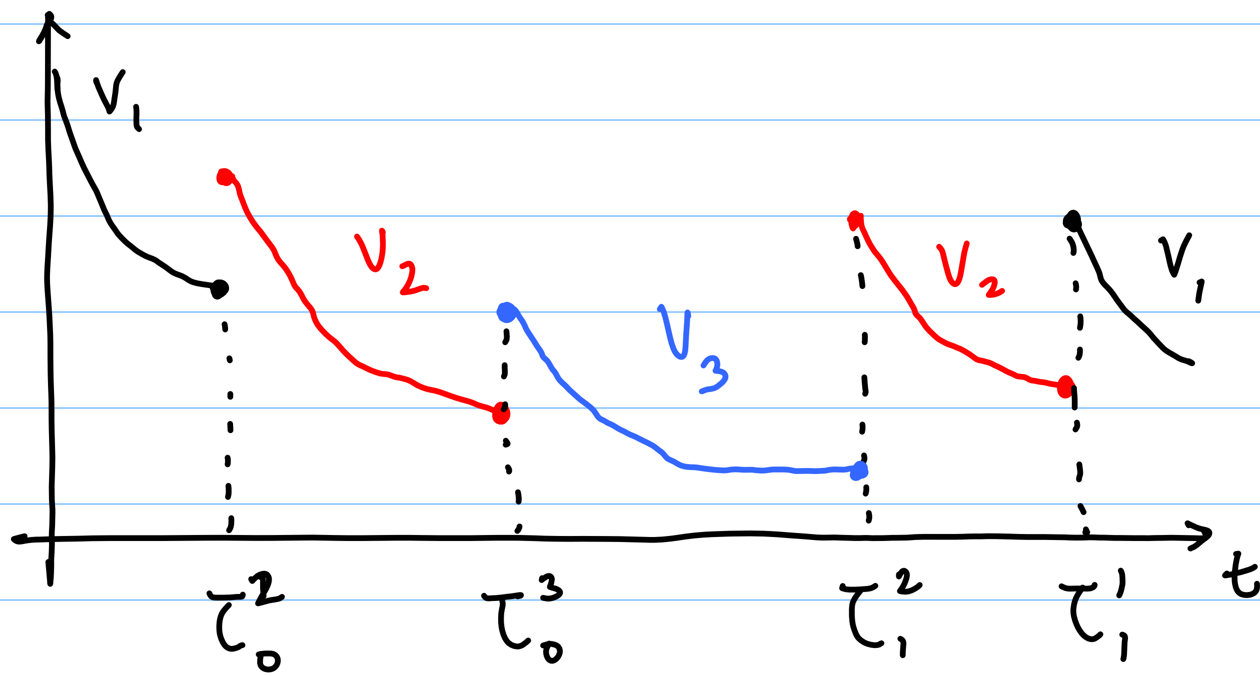
$$\dot{x} = f_i(x),$$

and let $p(t)$ define the switching. Let $\tau^i = \{\tau_0^i, \tau_1^i, \tau_2^i, \dots\}$ be the time instants, when the mode is switched to mode i.

Stability is guaranteed if for all $i \in \{1, 2, \dots m\}$, and $k \in \mathbb{N}$

$$\boxed{V_i(x(\tau_k^i)) \geq V_i(x(\tau_{k+1}^i))}$$

Illustration:



Every time a mode is entered, the energy of the mode is less than that when the mode is entered the previous time.

Switching of unstable systems

Consider a family of linear systems:

$$\dot{x} = A_{p(t)}\, x \,, \quad p(t) \in \{1, 2, \dots m\}$$
$$x \in \mathbb{R}^n$$

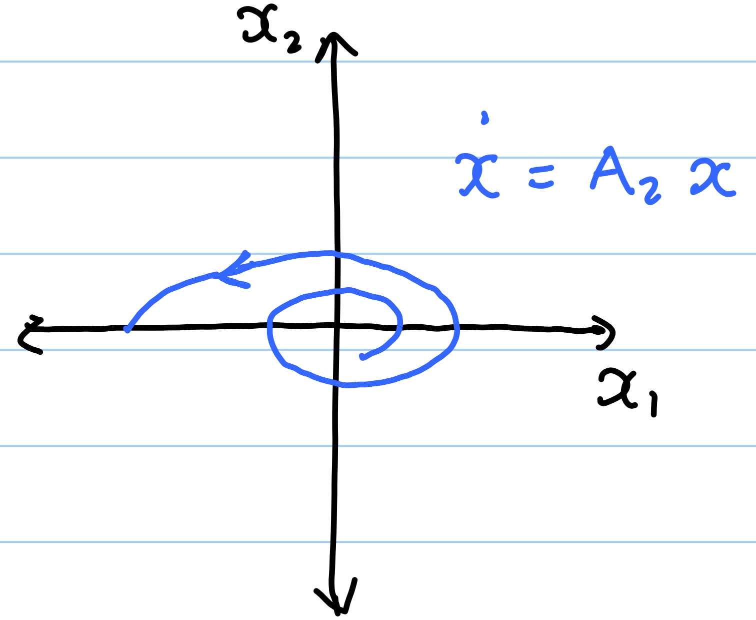
possibly with unstable modes.

If A is unstable, unless $A = \beta \cdot I$ for some $\beta \geq 0$
there is a positive definite P such that

$$x^T (PA + P^T A) x \leq 0$$

for some region $\Omega \in \mathbb{R}^n$

Example:



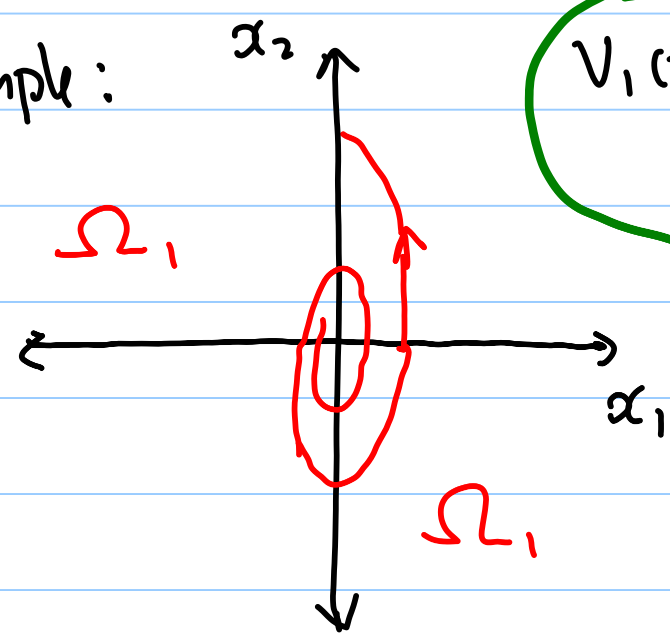
$\frac{d}{dt}(x^T x) \leq 0$, if $x_1 x_2 \geq 0$ (note: $P = I$)

The idea is to form a Lyapunov function by piecing together local Lyapunov functions.

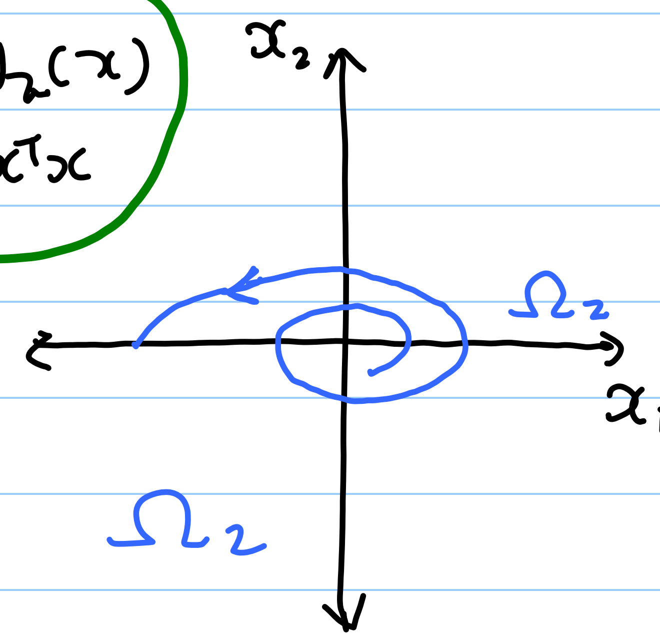
Stability is guaranteed if :

- $p(t) = i$ only if $x(t) \in \Omega_i$
- $V_i(x(\tau_k^i)) \geq V_i(x(\tau_{k+1}^i))$

Example :



$V_1(x) = V_2(x)$
$= x^T x$



Stabilization problem:

Choose the mode (design $p(t)$) so as to stabilize the system.

If there is a stable mode $\to$ trivial

All unstable mode, one possible strategy is to design $p(t)$ to minimize $x^T f_{p(t)}(x)$

Notice that: $x^T f_{p(t)}(x) = \frac{1}{2} \frac{d}{dt} \|x\|^2$,

Thus, we make the state converge to 0 in magnitude.

More systematic strategy for linear systems:

Let A_i, $i \in \{1, 2, \ldots, m\}$ be a family of unstable matrices. We aim to get quadratic stability, i.e. there exists a positive definite matrix P such that

$$V(x) = x^T P x$$

, and there exist $p(t)$ and $\varepsilon > 0$ such that

$$x^T \left(P A_{p(t)} + P^T A_{p(t)} \right) x < \underbrace{-\varepsilon \, x^T x}_{\text{always decreases}}$$

switching strategy

For $m=2$, such a switching strategy exists iff there is an $\alpha \in (0,1)$ such that

$$A_{eq} := \alpha A_1 + (1-\alpha) A_2 \quad \text{is stable}$$

How to design the switching strategy:

1 Compute P such that

$$P A_{eq} + P^T A_{eq} < 0$$

This is possible since A_{eq} is stable (Lyapunov eq)

2 Define $Q_1 = -(PA_1 + P^T A_1)$
$Q_2 = -(PA_2 + P^T A_2)$

Notice : $x^T Q_1 x = -\frac{\partial V}{\partial x} \cdot A_1 x$

$x^T Q_2 x = -\frac{\partial V}{\partial x} \cdot A_2 x$

Let $\Omega_i = \{ x \mid x^T Q_i x > 0 \}$, $i = 1, 2$

$\Omega_1 \cup \Omega_2 = \mathbb{R}^n$, $\Omega_1 \cap \Omega_2 \neq \emptyset$

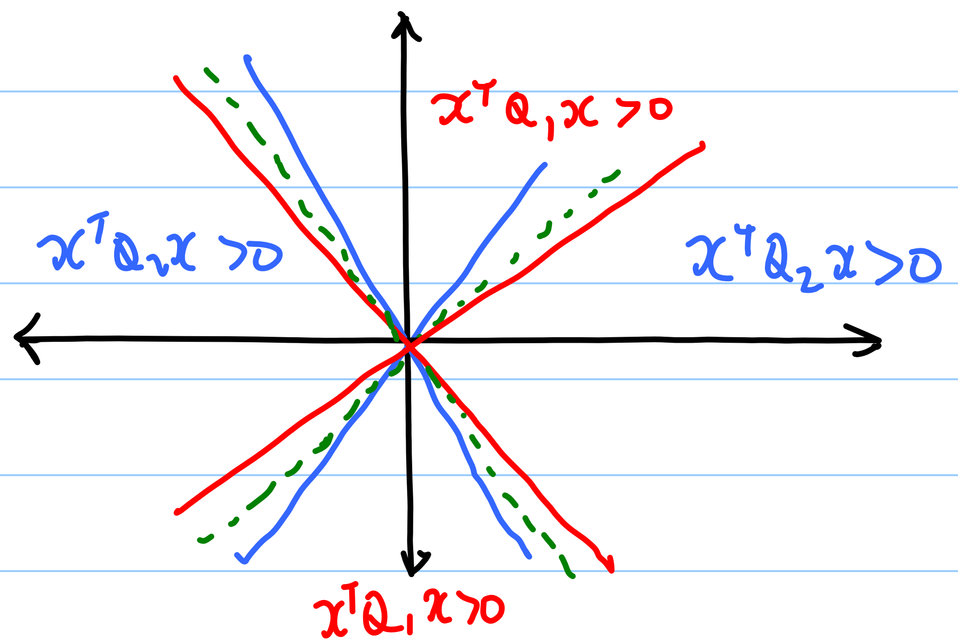
3 Define $s_1(x) = x^T (Q_1 - \varepsilon Q_2) x$
$s_2(x) = x^T (Q_2 - \varepsilon Q_1) x$

with ε positive small

4 Define switching strategy as follows:

* $P(0)$ is such that $x(0) \in \Omega_{P(0)}$
* $P(t^+) = 2$ if $P(t) = 1$ and $S_1(x) = 0$
* $P(t^+) = 1$ if $P(t) = 2$ and $S_2(x) = 0$

Analysis:



For $m > 2$, a sufficient condition is the existence of
$\alpha_i \in (0,1)$, $i \in \{1, 2, \dots m\}$ such that

$$\sum \alpha_i = 1, \text{ and}$$

$$A_{eq} := \sum \alpha_i A_i \quad \text{is stable}$$

Stabilization can then be done a la pulse width modulation control for sufficiently short period.