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Optimal Circadian Rhythm Control with Light Input for Rapid Entrainment and Improved Vigilance

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***Abstract*— Circadian rhythm is the biological process critical to the well being of all living organisms. The circadian rhythms oscillate with a period of approximately 24 hours due to the light-darkness pattern of the solar day. Circadian disruption, as experienced by night shift workers, travelers, submariners or miners, can lead to lower productivity, sleep disorder, and other more serious health problems. Using artificial light to regulate the circadian rhythm has long been proposed. The common approach is to use the phase response curve — the amount of steady state phase shift due to light pulses applied at specified times. In this paper, we consider a commonly used nonlinear second order oscillator model for the circadian rhythm response with light intensity as the input. Our first goal is to establish a performance bound by solving the minimum time control problem for a specified phase shift with contrained light intensity. The result is a much faster phase shift as compared to natural light-darkness pattern. We further extend the optimal control to vigilance, which is regulated in part by circadian rhythm, to maximize a vigilance lower bound for specified time and duration. Based on the two-process model of vigilance, the problem is formulated as an optimal control of switched system, and the optimization strategy is demonstrated via a simulation example.**

I. Introduction

Due to the 24-hour lighting-dark cycle on earth, circadian rhythm regulates the biochemical processes of almost all living organisms, including plants, insects, and mammals. Maintaining the regular cyclicity of this internal clock, called entrainment, is important to the well being of all organisms. For humans, circadian disruption can lead to lower productivity, digestive problems, decreased sleep efficiency and other health problems [1], [2]. The 24-hour pattern of light and dark is the strongest synchronizer of circadian rhythms to the solar day, and the human circadian system is most sensitive to blue light. Therefore, light stimuli may be used as treatment for circadian disruption. Commercial products such as Philips goLITE use self-administered blue LEDs to try to achieve better performance and better mood [3]. There have also been control-theoretic approaches to determine the timing and dose of the light stimulus to achieve the desired circadian state. A common approach is to use the Phase Response Curve (PRC), which relates the steady state phase shift of the circadian rhythm as a function of the time during a day when a light pulse with given amplitude and duration is applied. Light-dark pattern for jet lag treatment has been proposed [4] based on the PRC generated from an empirical nonlinear oscillator model for human circadian rhythm (the Kronauer model) [5]. A closed loop phase shift control based on PRC generated from *Drosophila melanogaster* gene network model is developed in [6]. The drawback of the PRC based method is that only the timing of the individual pulses can be adjusted because PRC does not capture how the duration and the intensity of the pulses influence the circadian rhythm.

A variety of biochemical models have been proposed to describe the circadian rhythm. Based on the study of *Drosophila melanogaster*, a 10-state biochemical model has been proposed in [7], [8]. More complex mammal biochemical model has been proposed in [9], but such complex model has not been used in control analysis. The output of these complex biochemical models mostly consists of concentration of chemicals in cells, which is difficult to measure continuously for control applications. An alternative approach of modeling the circadian rhythm is phenomenological models [5], [10]. The Kronauer model [5] uses core body temperature (CBT) as the circadian rhythm marker, and captures the essential behavior of the human CBT oscillation and the effect of light on the phase and amplitude of this oscillation. As demonstrated in [11], the model may be considered as the asymptotic case of the biochemical model in an averaged sense. One advantage of the model is that the output of the model, CBT, may be estimated continuously. This model has been used for model-based estimation and control [12].

The feedback entrainment control strategy based on Kronauer's model has been proposed in our previous work [13]. In simulation, jet lag recovery can be shortened using reference tracking control. In this paper, we establish the performance bound by formulating the entrainment process as a minimum time control problem. A novel shooting method is developed to solve the two-point boundary value problem (TPBVP) from the Pontryagon Minimum Principle. We compare the entrainment time for three different strategies: natural light-dark pattern, our previous feedback strategy in [13], and the minimum time control in this paper.

We also show our initial results in extending lighting control beyond circadian rhythm entrainment towards vigilance enhancement. Experimental data in the literature have indicated that circadian rhythm also contributes to the vigilance level. The Kronauer model is the basic components of the two-process model for human vigilance [14], [15]. By using

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the proposed shooting method, we extend the optimal control to maximize the lower bound of vigilance level for specified time and duration (e.g., for examination, military mission, etc.) by controlling light intensity and sleep timing.

II. Minimum time entrainment optimal control

A. Problem Formulation

Models of varying complexity have been proposed for circadian rhythm oscillation. We use the Kronauer's model [5] which is a phenomenological model consisting of a second order nonlinear oscillator with a 24.2-hour period, called the *P*-process, driven by a photoreceptor model, called the *L*-process. The *L*-process converts the light stimuli to a drive u, analogous to the bleaching of photopigments in retinal photoreceptor by photons. The photopigments remain unusable for further photon response until they are regenerated. A simple population model is used in [5]:

$$\begin{aligned} \dot{n} &= 60[\alpha(n-1) - \beta n], \\ u &= G\alpha(1-n) \end{aligned} \tag{1}$$

where $\alpha = \alpha_0(\frac{I}{I_0})^q, \alpha_0 = 0.16\,\text{min}^{-1}, \beta = 0.013\,\text{min}^{-1}, G = 19.9, q = 0.6, I_0 = 9500$ lux.

For a fixed intensity light input I, $n \to \alpha/(\alpha+\beta)$ with rate $60(\alpha+\beta)$. Comparing with the circadian dynamics which has a 24.2-hour period, the L process is much faster. In this paper, we assume that the *L*-process is sufficiently fast so that we can just focus on the dynamics of the *P*-process. As n converges to a steady state, the corresponding $u \to G\alpha\beta/(\alpha+\beta)$, which is a monotonically increasing function in α. We fix the maximum light intensity that can be used in the light treatment $I_{\max} = 9500$ lux. The corresponding maximum drive is $u_{\max} = 0.2392$. Light intensity cannot be negative, so the input bound is $0 \le u \le u_{\max}$. The drive u feeds into the *P*-process which is a nonlinear oscillator representing the circadian pacemaker. The model for the *P*-process is given by [5] (the unit of time is hour):

$$\dot{z} = f_0(z) + f_1(z)u. \tag{2}$$

where

$$\begin{aligned} z &= \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \; f_0(z) = Az + Bg(B^T z) \\ A &= \frac{\pi}{12}\begin{bmatrix} \mu/3 & 1 \\ -(24/(.99729\tau_x))^2 & 0 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ g(x) &= \frac{\pi}{12}\mu\left(\frac{4}{3}x_1^3 - \frac{256}{105}x_1^7\right) \\ f_1(z) &= \frac{\pi}{12}\begin{bmatrix} 1 & qx_2 + kx_1 \end{bmatrix}^T (1-0.4x_1)(1-0.4x_2). \end{aligned}$$

The state x_1 corresponds to the CBT variation which is used as the phase marker of the circadian system.

For rapid entrainment, we aim to establish a lower bound on how fast light control can shift the circadian cycle by a specified phase. A motivating example is a traveler traveling through multiple time zones. Upon arrival at the destination, the traveler's circadian rhythm will be different from that of the local population who are entrained to the local time. Our goal is to find out the light pattern which can help the traveler's circadian rhythm catch up with the local circadian rhythm rapidly. We pose this problem as a minimum time control problem. Denote the traveler's circadian rhythm z as in (2), and the local circadian rhythm z_r as the reference. The dynamics of z is given by (2) and z_r is given by:

$$\dot{z}_r = f_0(z_r) + f_1(z_r)u_{natural} \tag{3}$$

where $u_{natural}$ is the local light cycle modeled as a 12-hour light (9500 lux) and 12-hour dark pattern:

$$u_{natural} = \frac{u_{\max}}{2}(1 + \text{sgn}(\sin(\frac{\pi}{12}(t+t_0)))).$$

If the local light profile is known, it may be used instead. The time shift t_0 is determined by the initial temporal phase difference between z and z_r. The goal is to find $u(t)$, $0 \le u(t) \le u_{\max}$ to minimize the time for z to catch up with z_r:

$$J = t_f = \int_0^{t_f} 1\,dt. \tag{4}$$

where t_f is the final time to be determined as part of the solution. The boundary conditions are:

$$\text{Initial condition}: z(0) = z_0,\, z_r(0) = z_{r0}, \tag{5}$$

$$\text{Terminal condition}: z_r(t_f) = z(t_f). \tag{6}$$

where $z(0)$ and $z_r(0)$ are specified.

B. Solution of the Minimum Time Control Problem

The necessary condition for the minimum time control problem can be readily stated using the Pontryagin Minimum Principle [16]. The Hamiltonian of the system is

$$H = 1 + p^T\dot{z} = 1 + p^T(f_0(z) + f_1(z)u). \tag{7}$$

where p is the Lagrange multiplier for the constraint of the system dynamics, called co-state and satisfies

$$\dot{p} = -\frac{\partial H}{\partial z} = \underbrace{-\frac{\partial(f_0(z) + f_1(z)u)^T}{\partial z}}_{f_2(z,u)} p. \tag{8}$$

The optimal control u^* minimizes H and is given by the following "bang-off" control (assuming no singular arcs):

$$u^* = \frac{u_{\max}}{2}(1 - \text{sgn}(p^T f_1(z)). \tag{9}$$

Since t_f is free, the time varying (through $z_r(t_f)$) terminal constraint leads to the transversality condition:

$$1 + p^T(t_f)(\dot{z}(t_f) - \dot{z}_r(t_f)) = 0. \tag{10}$$

Substituting the optimal control into the state and co-state equations, we obtain the following two-point boundary value problem (TPBVP):

$$\begin{aligned} \dot{z} &= f_0(z) + f_1(z)\frac{u_{\max}}{2}(1 - \text{sgn}(p^T f_1(z)), & (11) \\ \dot{p} &= f_2(z, \frac{u_{\max}}{2}(1 - \text{sgn}(p^T f_1(z)))p. & (12) \end{aligned}$$

where $z(0) = z_0$ is specified, the terminal state is given by $z(t_f) = z_r(t_f)$ with z_r satisfies (3), and t_f satisfies the transversality condition (10).

There are multiple techniques to solve the TPBVP numerically. One may regard the problem as having $n+1 = 3$ unknowns, $(p(0), t_f)$, and three algebraic equations, the terminal state constraint (6) and transversality condition (10). Standard nonlinear programming falls into local minima depending on the initial guess, however it is very hard to exhaust all the initial guesses in $\mathbb{R}^3$ to find the smallest local minimum. Standard boundary value problem solver runs into numerical difficulty due to the unstable co-state equation propagation. Another approach is to convert the free terminal time problem to the fixed terminal time through the normalization $\tau = t/t_f$. This results in an order $2n+1 = 5$ TPBVP. Direct numerical solution technique such as multiple shooting method [17] also requires a reasonably good initial guess of the state trajectory, which is difficult to obtain in general. Other approaches such as direct update of the switch times (similar to [18]) and continuation method (starting from the solution for a linear oscillator and then propagating along the parameter to add in the nonlinear terms) have also been attempted but with only limited success.

We propose an alternative approach in this paper, by first making the following observation about the TPBVP (without the transversality condition):

Proposition 1: If $(z(t), p(t))$ is a solution of (11)-(12), then so is $(z(t), \alpha p(t))$ for any positive constant α.

Proof: Let $(z(t), p(t))$ be a solution of (11)-(12). For any $\alpha > 0$,

$$\frac{u_{\max}}{2}(1 - \mathrm{sgn}(p^T f_1(z)) = \frac{u_{\max}}{2}(1 - \mathrm{sgn}(\alpha p^T f_1(z)).$$

Hence z also satisfies

$$\dot{z} = f_0(z) + f_1(z)\frac{u_{\max}}{2}(1 - \mathrm{sgn}(\alpha p^T f_1(z)),\ z(0) = z_0.$$

Now scale both sides of (12) by α, we get

$$\alpha \dot{p} = f_2(z, \frac{u_{\max}}{2}(1 - \mathrm{sgn}(\alpha p^T f_1(z)))\alpha p.$$

which is the same as (12) with p replaced by αp. Hence (11)–(12) are satisfied by $(z(t), \alpha p(t))$. □

Based on the above Proposition, we note that if the initial co-state $p(0)$ is scaled by a positive constant α, the trajectory of $z(t)$ will not change, and the co-state trajectory $p(t)$ will also be scaled by α. Since $p(0) = [p_1(0), p_2(0)]^T$ is a 2×1 vector, we can parameterize it in the polar coordinate as $p(0) = \alpha[\cos(\phi), \sin(\phi)]^T$, and only the angle $\phi = \tan^{-1}(\frac{p_2(0)}{p_1(0)})$ affects the trajectory of $z(t)$.

This observation leads to the following strategy for solving the TPBVP:

1. Search $\phi \in [0, 2\pi)$ in $p(0) = [\cos(\phi), \sin(\phi)]^T$ to satisfy the state terminal constraint (6). The numerical threshold $\|z(t) - z_r(t)\| < \varepsilon$ will be used as the termination condition.
2. Let ϕ^* be the angle from step 1 that corresponds to the smallest t_f and satisfies $p(t_f)^T(\dot{z}(t_f) - \dot{z}_r(t_f)) \neq 0$.
3. Choose $\alpha = -(p(t_f)^T(\dot{z}(t_f) - \dot{z}_r(t_f)))^{-1}$ to satisfy the transversality condition.

C. Minimum Time Circadian Rhythm Entrainment

In this section, we consider the entrainment for a specified phase shift. The initial condition of the controlled oscillator is set to $z(0) = [1.06, 0]^T$ which is on the limit cycle of the circadian rhythm oscillator entrained to natural light pattern. The initial condition of the reference oscillator $z_r(0)$ is also on the limit cycle, but it has a temporal phase difference from $z(0)$. The convergence criterion ε is arbitrarily chosen as 0.1. The upper bound for the minimum time is set at $t_{f_{\max}} = 200$hr ($8\frac{1}{3}$ days) since we know that natural entrainment for 12 hour phase shift is about 7 days. The plots of $\min_{t\in[0,t_{f_{\max}}]} \|z(t) - z_r(t)\|$ and the corresponding time $t_\phi = \arg\min_{t\in[0,t_{f_{\max}}]} \|z(t) - z_r(t)\|$ versus the co-state angle ϕ for a 6-hour phase shift are shown in Figure 1.

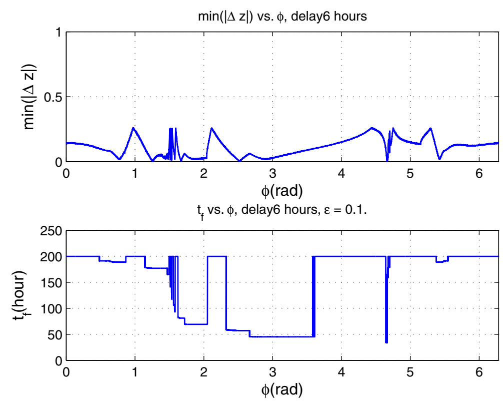

Fig. 1. Terminal state constraint and convergence time vs. co-state angle for 6 hour phase shifts.

For the entrainment to a specified phase shift, we compare the minimum time solution with two other methods:

1) The natural 12-hr light-dark cycle entrainment,
2) The feedback algorithm that we have previously presented [13]: $u = \frac{u_{\max}}{2}(1 + \mathrm{sgn}((z_r - z)^T f_1(z)))$.

From the comparison plot shown in Figure 2, we make the following observations:

- The minimum time solution is significantly better than the natural entrainment. For the 12-hour phase shift, natural entrainment takes about 7 days and optimal entrainment takes about $2\frac{1}{2}$ days. For the worst case 16-hour phase delay (or 8-hour phase advance), the difference is the largest: 10 days vs. 3 days.
- The entrainment time is larger for phase advance (going in the same direction as earth rotation), than for phase delay (in the opposite direction as earth rotation). This is due to the fact that the light input is more efficient in delaying the phase of human circadian rhythm than advancing the phase, which is captured in the Kronauer model.
- The feedback algorithm performs at similar level as the optimal control for phase delay, but worse for phase advance (but still better than the natural light-dark cycle entrainment).

The optimal light input, the corresponding state trajectories, and the local light pattern for different specified phase delays are shown in Figure 3. Note that light input may be required at inconvenient hours such as early in the morning and light (blue wavelength) may need to be blocked during the local day time.

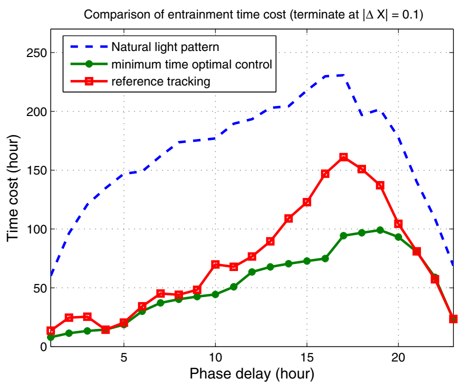

Fig. 2. Comparison of the time cost as a function of the specified phase shift under minimum time control, feedback control and natural light pattern.

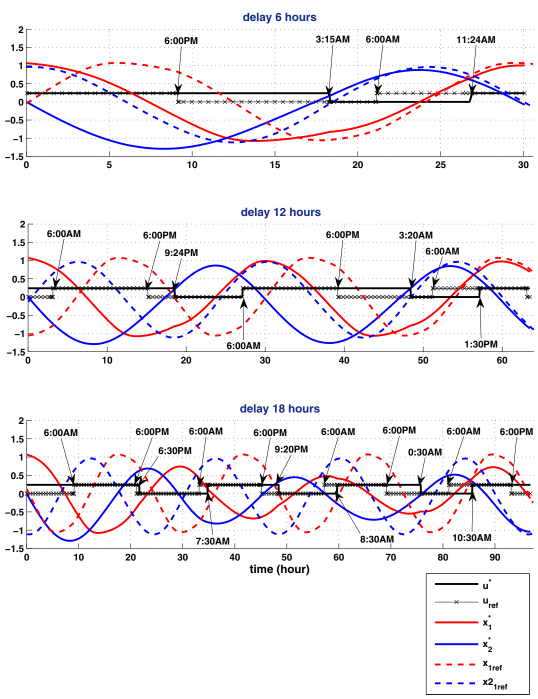

Fig. 3. Minimum time control and state trajectories for 6 hour, 12 hour, and 18 hour phase shifts, compared with the natural light pattern.

III. Extending to Human Vigilance Control

A. Problem formulation

Quantified human alertness may be measured by subjective alertness rating, psychomotor vigilance test (PVT), cognitive throughput test, Electroencephalography (EEG) and Electrooculography (EOG). Experimental data indicate that human vigilance is related to circadian rhythm and sleep homeostasis [14] [15] [19]. Among the seven biomathematical human vigilance models reviewed in [20], the model proposed by Achermann [14] is the only one that integrates circadian rhythm with vigilance and provides the thresholds for spontaneous sleeping and waking. We therefore choose the Achermann model as the basis for our vigilance optimal control study. The Achermann vigilance model combines the sleep homeostasis, sleep inertia and circadian rhythm. The circadian rhythm dynamics is based on the model proposed by Kronauer in 1990 [21]. This model has since been revised and refined in 1998 and 1999, so we replace the circadian dynamics in the Achermann model with the refined circadian dynamics in (2).

The modified Achermann vigilance model is of the following form:

$$\dot{z} = f_0(z) + f_1(z)(1-\beta)u, \tag{13a}$$

$$\dot{H} = -\beta H/\tau_d - (1-\beta)(H-1)/\tau_r, \tag{13b}$$

$$\dot{W} = -\frac{1-\beta}{\tau_W} W, \tag{13c}$$

$$B = H - A_c x_1 \tag{13d}$$

$$A = (1-\beta)(1-B-W) \tag{13e}$$

where $\beta(t)$ denotes the sleep state at time t: $\beta = 1$ if the person is asleep and $\beta = 0$ otherwise, (13a) is the circadian dynamics (same as (2)), H characterizes the sleep homeostasis, W is the sleep inertia (modeling the low vigilance just after waking up), B is sleepiness, and A is vigilance. Note that H, A, B are all normalized between 0 and 1. In Kronauer model, positive x_1 corresponds to circadian day and negative x_1 is the circadian night. Sleepiness, B, is therefore simply sleep homeostasis added with the circadian contribution. The model parameters are given by

$$\tau_d = 4.2\text{Hr},\ \ \tau_r = 18.2\text{Hr},\ \ \tau_W = .662\text{Hr}, A_c = 0.1333.$$

At the moment of waking, W is reset to 0.32. The sleep onset threshold is denoted by H_m, meaning that sleep can occur if $B > H_m$. The waking threshold is denoted by L_m, meaning that spontaneous waking occurs when $B < L_m$. The thresholds in the Achermann model are $H_m = 0.67$ and $L_m = 0.17$. We treat the sleep state as a variable that can be controlled, based on the state of sleepiness. If $B \geq H_m$, then the person falls asleep if allowed. If $B \leq L_m$ during sleep, then the person awakes spontaneously. However, if $B > L_m$ during sleep state, the person may still be awaken (e.g., by an alarm clock).

Global economy and military preparedness require best human performance during a mission. However, performance may be compromised if the mission time coincides with the endogenous nadir in vigilance. Examples include military missions scheduled during circadian night, multi time-zone travelers attending business meetings while suffering jet lag, and night-owl teenagers preparing for morning exams. For a mission scheduled during $[t_M, t_f]$, we assume $[t_0, t_M]$ is the preparation episode, with (t_0, t_M, t_f) all fixed. During the

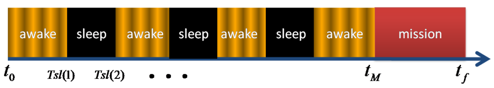

Fig. 4. Sleep cycle during mission preparation.

mission, the lighting is fixed to be $u_{mission}$ and the person is awake. The lighting $u(t)$ may be controlled whenever the person is awake during the preparation episode.

We pose the mission preparation problem as a max-min problem: maximizing the minimum vigilance during the mission by control the lighting $u(t)$ and sleep switching times $T_{\rm sl}$ during the preparation episode. As shown in Figure 4, $T_{\rm sl}$ consists an increasing sequence of time instants with an even number of elements. The odd element indicates the onset of sleep (β switching from 0 to 1), and even element indicates the termination of sleep (β switching from 1 to 0). We pose the max-min vigilance optimal control problem as follows:

$$J_{opt} \triangleq \max_{u(t), T_{\rm sl},\ t\in[t_0,t_M]} \min_{t\in[t_M,t_f]} A(t). \tag{14}$$

In addition to the physiological dynamics (13a, 13b and 13c), the boundary conditions and constraints include (assume K sleeping episodes):

$$z(t_0) = z_0 \tag{15a}$$

$$H(t_0) = H_0,\ W(t_0) = W_0 \tag{15b}$$

$$0 \le u(t) \le u_{\max} \tag{15c}$$

$$B(T_{{\rm sl}_{2i-1}}) \ge H_m,\ i = 1,\dots,K \tag{15d}$$

$$B(T_{{\rm sl}_{2i}}) \ge L_m,\ i = 1,\dots,K \tag{15e}$$

$$t_0 \le T_{{\rm sl}_1} \le \dots \le T_{{\rm sl}_{2K}} \le t_M. \tag{15f}$$

The initial state may be determined by the previous sleep-awake pattern and vigilance measurements. Note that when a person is schedule to sleep, the sleepiness measure B must be sufficiently high (greater than H_m), otherwise the person may not be able to fall asleep. The scheduled waking up should be earlier than the spontaneous waking (otherwise the person would have already be awake), so the sleepiness measure B at waking time must be greater than L_m.

B. Optimal control of switching system

We decompose the optimal control problem as two sub-problems:

- Subproblem 1: For a fixed T_{sl}, find the optimal light input $u(t), t \in [t_0, t_M]$ to maximize $\min_{t\in[t_M,t_f]} A(t)$ subject to (15a)–(15c):

$$J_1(T_{\rm sl}) \triangleq \max_{u(t),\, t\,\in\,[t_0,t_M]} \min_{t\in[t_M,t_f]} A(t). \tag{16}$$

- Subproblem 2: Maximize $J_1(T_{\rm sl})$ by selecting the sleep timing subject to the constraint (15d)–(15f):

$$J_{opt} = \max_{T_{\rm sl}} J_1(T_{\rm sl}). \tag{17}$$

Solution of Subproblem 1: During the mission, the person is awake; therefore, the sleep homeostasis converges to 1 and the sleep inertia decays to 0, both exponentially. The alertness during the mission is

$$A(t) = e^{-\frac{t-t_M}{\tau_r}}(1-H(t_M)) - e^{-\frac{t-t_M}{\tau_w}} W(t_M) + [A_c, 0]z(t). \tag{18}$$

In Subproblem 1, $T_{\rm sl}$, H_0 and W_0 are given, so $H(t_M)$ and $W(t_M)$ are determined by (13b) and (13c). The light input during the mission is constant, $u = u_{mission}$. For $t_M \le t \le t_f$, the state variables of the circadian rhythm $z(t)$ is determined by $z(t_M)$. The objective function $\min_{t\in[t_M,t_f]} A(t)$ is a function of $z(t_M)$ only, and may be computed from (18); we define it as $h(z(t_M))$. The optimization problem is then to find $\{u(t), t_0 \le t \le t_M\}$ to minimize $-h(z(t_M))$ subject to the initial condition and input constraint. The Hamiltonian of the system is given by

$$H = -h(z(t_M)) + p^T\dot{z} = -h(z(t_M)) + p^T(f_0(z) + f_1(z)u). \tag{19}$$

where p is the co-state. The optimal control and co-state equations are exactly the same as before, given by (8) and (9), respectively. The terminal condition for the co-state is given by

$$p(t_M) = -h'(z(t_M)). \tag{20}$$

Together with the initial state condition, we now have a fixed terminal time TPBVP. The equations of co-states are homogeneous, and the optimal control is "bang-off", so Proposition 1 again holds. Parameterize $p(t_0)$ as $\alpha[cos(\phi), sin(\phi)]^T$, then α does not change the state trajectory, $z(t)$, and only scale the co-state $p(t)$. To find (α,ϕ) to satisfy the terminal condition (20), we first search over the interval $\phi \in [0, 2\pi)$ so that $p(t_M)$ and $h'(z(t_M))$ are pointing in the opposite direction, i.e.,

$$p(t_M)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} h'(z(t_M)) = 0,\ p^T(t_M)h'(z(t_M)) < 0. \tag{21}$$

Once ϕ is found, α is then chosen to scale $p(t_M)$ to the same magnitude as $h'(z(t_M))$. Note that $h'(z(t_M))$ will need to be computed numerically. Finally, the optimal light input for given $T_{\rm sl}$ is given by (9).

Solution of Subproblem 2: Subproblem 2 involves finding the sleep times $T_{\rm sl}$ subject to the constraints (15d, 15e and 15f) to maximize $J_1(T_{\rm sl})$. We use the gradient descent to adjust the sleep times, with the gradient computed numerically. The algorithm is described below:

- Initial guess of $T_{\rm sl}$: We choose a sufficiently large number of short sleep episodes to guarantee sufficiently high homeostasis so that (15d)–(15e) are satisfied.
- Update the $T_{\rm sl}$ by following the descent direction of the numerical gradient. Project the gradient to ensure the sleepiness constraint is not violated. The numerical gradient is needed for ∇J_1 and $\nabla B(T_{{\rm sl}_i})$, $i = 1 \dots 2K$ with respect to $T_{\rm sl}$.

As a simulation example, let $t_M = 60$ hr, $t_f = 70$ hr, and there are two sleep episodes during the preparation episode ($K = 2$), the optimization result is shown in Figure 5. As

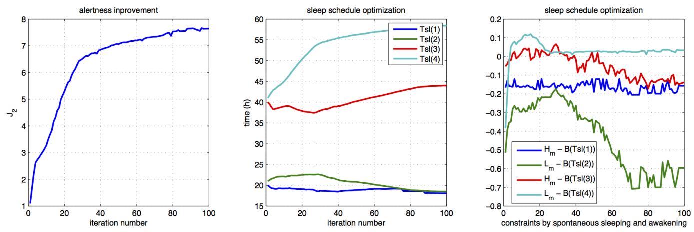

Fig. 5. Sleep schedule optimization with 100 iterations. Left: the vigilance lower bound in mission vs. optimization iteration. Middle: the sleep schedule vs. optimization iteration. Right: Sleepiness constraints during optimization.

the iteration number increases, the duration of the fist sleep episode is compressed to almost zero and the beginning of the second sleep episode is postponed. The sleep deficit is so high that the second sleep episode is extended and consolidated. The constraint (15d) is not violated; the person is sleepy enough at the sleep onset and wakes up before the sleepiness drops below the spontaneous waking up threshold. However the optimized sleep schedule is uncomfortable because the person needs to stay awake for an extended period. In the future, we plan to take into account factors such as comfort and sleep efficiency, as well as more complex mission schedules such as rotating shift work.

IV. CONCLUSIONS

This paper considers the optimal control for rapid circadian rhythm entrainment and vigilance improvement. From the analysis of the nonlinear circadian rhythm model, the necessary conditions for the minimum time tracking problem are obtained. A modified shooting method is presented to solve the nonlinear TPBVP. We demonstrate this method on the minimum time circadian phase shift problem, and compare it with our previous feedback method and the natural light-dark pattern. Utilizing a two-process human vigilance model, we also extended the modified shooting method to the optimal control of lighting and sleep timing for improved vigilance during mission.

V. ACKNOWLEDGMENTS

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