

Online Algorithm for Leasing Wireless Channels in a Three-Tier Spectrum Sharing Framework

Gourav Saha, Alhussein A. Abouzeid, and Marja Matinmikko-Blue

***Abstract*—The three-tier spectrum sharing framework (3-TSF) is a spectrum sharing model adopted by the Federal Communications Commission. According to this model, under-utilized federal spectrum like the Citizens Broadband Radio Service band is released for shared use where the highest preference is given to Tier-1 followed by Tier-2 (T2) and then Tier-3 (T3). In this paper, we study how a wireless operator, who is interested in maximizing its profit, can strategically operate as a T2 and/or a T3 user. T2 is characterized by paid but "almost" guaranteed and interference-free channel access while T3 access is free but has the lesser guarantee and also faces channel interference. So the operator has to optimally decide between paid but better channel quality and free but uncertain channel quality. Also, the operator has to make these decisions without knowing future market variables like customer demand or channel availability. The main contribution of this paper is a deterministic online algorithm for leasing channels that has finite competitive ratio, low time complexity, and that does not rely on the knowledge of market statistics. Such algorithms are desirable in the early stages of the deployment of 3-TSF because the knowledge of market statistics may be rather inaccurate. We use tools from the ski-rental literature to design the online algorithm. The online optimization problem for leasing channels is a novel generalization of the ski-rental problem. We, therefore, make fundamental contributions to the ski-rental literature, the applications of which extend beyond this paper. We also conduct simulations using synthetic traces to compare our online algorithm with the benchmark and state-of-the-art algorithms.**

***Index Terms*—CBRS band, spectrum sharing, spectrum licenses, opportunistic spectrum access, online algorithms, ski-rental problem, competitive ratio**

I. Introduction

The demand for wireless Internet access is ever growing and there is a notion that the wireless spectrum is getting scarce. The President's Council of Advisors on Science and Technology (PCAST) called the notion of spectrum scarcity a *"fundamental misunderstanding"* [1] arising due to under-utilization of spectrum. In support of the PCAST report [1], the FCC decided to release the underutilized Citizens Broadband Radio Service (CBRS) band for shared use [2] and finalized the rules in [3]. CBRS band is a 150 MHz wide federal spectrum band spanning $3.55 - 3.7$ GHz used primarily for US government radar systems. The shared use of CBRS band follows the *Three-Tier Spectrum Sharing Framework* (3-TSF) as shown in Figure 1: *Tier-1* (T1), also called the "Incumbent tier," consists of federal users who have the highest priority access to any channel and are guaranteed interference protection from lower tiers. *Tier-2* (T2) is called the "Priority Access Licenses (PAL) tier." T2 users can lease the channels by participating in auctions which happen periodically in time duration of years; currently it is three years [3]. The contract duration of a channel lease is also three years after which it is again put to auction. T2 users can use the leased channels whenever T1 users are not using it. They are guaranteed interference protection from Tier-3 users. *Tier-3* (T3) is called the "Generalized Authorized Access (GAA) tier." T3 users can opportunistically use a channel for *free* provided that it is not used by T1 or T2 users. A T3 user is not guaranteed interference protection from T1, T2 or even other T3 users. The number of opportunistic channels available for T3 users can change in a time scale which is much smaller[1] compared to T2 which operates in time scale of years. The Spectrum Access System (SAS) is a central database which keeps record of channel states [2]. It is also a policy engine which enforces the three tier hierarchy.

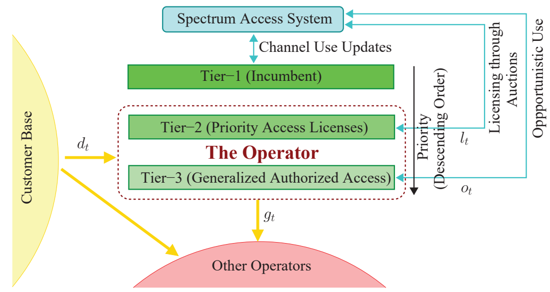

Figure 1. The Three-Tier Spectrum Sharing Framework and our System Model.

In this paper, we study how to maximize the profit of a wireless operator, which serves customer demand by using shared channels governed by 3-TSF. We consider a time slotted model. In every time slot, the wireless operator has to decide the amount of customer demand to reject, the amount of customer demand to serve using opportunistic channels and the number of channels to lease. A channel lease has a contract duration of years while a time slot ranges from minutes to days. Therefore, a leased channel can be used to serve the demand of the current as well as the future time

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This work has supplementary downloadable material available at http://ieeexplore.ieee.org.

[1]Our conversations with experts suggests that the number of opportunistic channels can change in time scale of minutes to days.

slots. Given that the cost of leasing a channel is substantial, possibly millions of dollars, the benefit of leasing a channel relies on future demand, channel availability, etc. The operator has to make the decisions without the future knowledge of these variables. The online nature of the problem leads to the following uncertainties when the operator wants to lease a channel: 1) *Uncertainty in demand:* Leasing a channel is profitable if the demand is high in the future time slots. 2) *Uncertainty in availability and quality of opportunistic channels:* Leasing a channel is profitable if there are not enough channels for opportunistic use in future time slots or if the opportunistic channels have high interference. 3) *Uncertainty in channel availability for leasing:* Other operators may lease all the channels in the future time slots. It is also possible that the operator does not win channel leases in future auctions. Therefore, it may be profitable to lease channels in the current time slot. 4) *Uncertainty in T1 channel usage:* Leasing channels is profitable if T1 usage is low in the future time slots. The operator may also have to lease additional channels to compensate for those channels which gets preempted by T1 usage in the future time slots. 5) *Uncertainty in service price:* Service price is the operator's income for serving a unit of customer demand. If service price is high in future time slots, then rejecting customer demand will lead to higher losses. Therefore, it is better to lease channels to serve customer demand. Due to these uncertainties, maximizing the profit of the wireless operator using 3-TSF is a challenging online optimization problem which we address in this paper.

Our online optimization problem has striking resemblance with ski-rental problem (SRP). In SRP, a skier has to decide between renting or buying a pair of skis without knowing the number of days he/she will be skiing. In our problem, buying skis is equivalent to leasing a channel while renting skis is equivalent to rejecting demand and serving the accepted demand using opportunistic channels. The unknown number of ski days can be mapped to uncertainty in demand and opportunistic channel availability. Using this analogy, we design a deterministic online algorithm for leasing channels which has finite competitive ratio. Despite many similarities with SRP, our problem has a distinct feature not found in other ski-rental literature; the uncertainty in channel availability for leasing. Due to this uncertainty the operator may have to wait to lease a channel. In order to get practically viable competitive ratio, we upper bound the *wait time*, the time difference between when an operator *decides* to lease a channel to when it leases a channel. This is achieved by lower bounding the average channel availability for leasing (assumption A3). Higher average channel availability implies lower wait time.

In terms of related work, [4] has a lot of resemblance with our work. In [4], the authors modeled the demand and channel availability statistics as a discrete time markov chain. It then used tools from stochastic dynamic programming to design an online algorithm for leasing channels. Their online algorithm has pseudo-polynomial time complexity if the optimization horizon is greater than the lease duration. It should be noted that resemblance between our work and [4] only exists in the mathematical abstraction of the problem. However, our problem statements are different. Problems similar to [4] have been addressed in [5], [6]. In [5], a network operator in a non-cooperative market has to optimally decide the portfolio of dedicated spectrum (equivalent to leasing) and shared spectrum (equivalent to opportunistic use) to maximize the expected profit. A similar problem is addressed in [6] but from the perspective of risk-averse, risk-neutral and risk-seeking wireless operators. These works [4], [5], [6] assume knowledge of market statistics. There are other bodies of work that are of importance to the 3-TSF. In [7], the authors designed a network protocol and an SAS which implements the rules of the 3-TSF. The work done in [8] considers a market where an operator can operate in either T2 or T3. It investigates the incentive of an operator to enter such a market in presence of competition. Other areas of research can be of significance to the 3-TSF though they are not directly related. From an economic standpoint, research in the field of spectrum contracts [9], [10], auctions and pricing [11] help to understand if the 3-TSF is economically attractive for potential investors. From a technical standpoint, dynamic channel allocation is of significant importance to 3-TSF. It is crucial to consider blocking probability [12] and co/adjacent channel interference [13] while doing dynamic channel allocation.

We now present an overall outline of the paper. We start by presenting the system model in Section II-A. In our system model, the operator can serve customer demand by operating as T2 and/or a T3 user. In order to maximize its profit, the operator has to strategically operate as T2 and/or a T3 user. This is mathematically captured using optimization problem $OP1$ formulated in Section II-B. $OP1$ does not provide much insight as to how we can solve the problem online. In this regard, we derive Theorem 1 in Section II-D which decouples $OP1$ into two optimization problems $OP2$ and $OP3$. The optimal solution of $OP2$ can be found using only online information and using standard algorithms. However, we need offline information to find the optimal solution of $OP3$. Since offline information is not available in practice, we find a deterministic online algorithm to solve $OP3$ as follows. *First*, we note that $OP3$ has strong resemblance with the optimization problem considered in [14]. In [14], the authors leveraged the Bahncard Problem, a variant of SRP, to design their online algorithm. This inspires us to relate $OP3$ to SRP in Section III-A. We show that $OP3$ can be reduced to a modifed version of SRP called MSRP, where MSRP is SRP in the presence of wait time. We design a deterministic online algorithm for MSRP in Section III-B and prove that it has an optimal competitive ratio. *Second*, we draw insight from the study of MSRP to design a deterministic online algorithm for $OP3$, and hence $OP1$, in Section III-C. We also derive its competitive ratio in Theorem 4 and time complexity in Theorem 5. In Section IV, we present simulations using sysnthetic traces to compare our online algorithm with other benchmark algorithms. These simulations reveal useful trends concerning the performance of our online algorithm. Finally we conclude the paper in Section V with a brief discussion of the immediate extensions to this work.

The main contributions of this paper are as follows. *First*, our system model is novel as it captures key elements of 3-

TSF such as the three-tier hierarchy and the low QoS associated with using opportunistic channels. *Second*, we design a deterministic online algorithm for MSRP that is optimal in the sense of competitive ratio. This algorithm is a non-trivial extension of the conventional *breakeven algorithm* for SRP. For example, while the conventional breakeven algorithm is based on one threshold, the online algorithm for MSRP is based on two thresholds. The study of MSRP constitutes the principal theoretical contribution of the paper which may have applications beyond the problem considered in this paper. *Third*, our online algorithm for $OP1$ does not require statistical knowledge of the involved random processes like demand and channel availability. Such algorithms will be desirable in the early stages of deployment of 3-TSF because the knowledge of market statistics will be rather inaccurate or completely unknown. Also, our algorithm has a polynomial time complexity irrespective of the optimization horizon. *Fourth*, this paper adds a new application area to the SRP. In the past, SRP inspired online algorithm designs for TCP acknowledgement [15], cloud computing [14], data center power optimization [16] and automobile idling [17].

II. Problem Formulation

In this section we first propose our system model which captures key elements of the 3-TSF. We then formulate optimization problem $OP1$ which is a generalization of the profit maximization problem of the operator. The underlying assumptions in our problem formulation is listed next. We end this section by introducing Theorem 1 which helps in the following ways. *First,* it effectively reduces the number of decision variables from three to two. *Second,* it provides a quantitative framework to understand the online nature of $OP1$. *Third,* it lays the groundwork which helps us relate our optimization problem with SRP in Section III-A.

A. System Model

There is a market consisting of many operators. A total of M channels are released for shared use following the 3-TSF. The operators use these channels to serve customer demand. One such operator, labelled “The Operator” is shown in Figure 1. The objective is to *maximize the profit* of the operator. In our model, the operator can work as T2 and/or T3 user. We consider a time slotted model where a time slot, also called epoch, may range from minutes to days. In every epoch $t \in \{1, \dots, T\}$, the operator receives d_t demand from the customers. Customer demand is assumed to be a discrete variable which can be expressed in bits per second (bps), e.g., one unit of customer demand equals $5\,kbps$. Our model also considers that PAL auctions can be conducted in every epoch, i.e. real time auctions. This is a deviation from the current 3-TSF in which PAL auctions are conducted every three years [3]. However, it is plausible to envision real time auctions in the near future. Upon receiving the demand, the operator has to make the following decisions in every epoch:

1) Amount of demand to reject. This is denoted by g_t. The operator accepts to serve $(d_t - g_t)$ demand using either opportunistic channels and/or leased channels.

Table I
A table of frequently used notations.

Notation	Description
t	t^{th} epoch.
T	Optimization horizon.
M	Total number of channels.
$\mathcal{H}$	Spectral efficiency of a channel. It is defined as the amount of demand which can be served per channel.
d_t	Amount of demand at epoch t.
d_M	Upper bound on d_t, $d_t \leq d_M$.
g_t	Amount of demand the operator rejected at epoch t.
p_t	Operator's income per unit demand served at epoch t.
p_M	Upper bound on $\mathcal{H}p_t$, the maximum revenue which the operator can earn per channel at epoch t. Mathematically, $\mathcal{H}p_t \leq p_M$.
τ	Contract duration of a channel lease.
P	Price that the operator pays per channel lease. In general, $p_M \ll P < \tau p_M$.
M_t^l	Number of channels available for leasing at epoch t.
l_t	Number of channels the operator leases at epoch t.
v_t	Total number of channels that all the other operators leases at epoch t
W_t	Total number of channels that all the operators in the market lease at epoch t.
M_t^o	Number of channels the operator *can* opportunistically use at epoch t.
o_t	Amount of demand served using opportunistically available channels at epoch t
A_t	Number of active channel leases the operator has at epoch t.
λ_t	Number of active channel leases of the operator that got pre-empted by T1 users at epoch t, $\lambda_t \leq A_t$.
$f_t\left(x\right)$	A function to penalize opportunistic channel use.
φ_t	$\varphi_t = (d_t\,, \lambda_t\,, p_t\,, M_t^o\,, v_t\,, f_t\left(\cdot\right))$, a tuple which forms the input to optimization problem $OP1$.
μ_l	Lower bound on the moving average of M_t^l over τ epochs, $\mu_l \leq \frac{\sum_{t=t_o-\tau+1}^{t_o} M_t^l}{\tau}$; $\forall t_o$.
r_t	It implies renting. Mathematically, $r_t = g_t + o_t$, the sum of rejecting g_t demand and serving o_t accepted demand using opportunistic channels.
$F_t\left(x\right)$	Renting function: A function to penalize the renting action.
D_t	Effective demand, $D_t = d_t + \mathcal{H}\lambda_t$.
ψ_t	$\psi_t = (D_t\,, v_t\,, F_t\left(\cdot\right))$, a tuple which forms the input to optimization problem $OP3$.
η	Wait time to purchase one or many channel/ski leases.
η_M	Upper bound on wait time η, $\eta \leq \eta_M$.
$(x)^+$	Positivity operator: $(x)^+ = \max\left(0, x\right)$
$\mathbb{Z}_+$	Set of non-negative integers.

2) Amount of accepted demand to serve using opportunistic channels. This is denoted by o_t, where $o_t \leq d_t - g_t$. In doing so, the operator behaves as a T3 user. The operator serves $(d_t - g_t - o_t)$ demand using leased channels.
3) Number of channels to lease in order to serve the accepted demand. This is denoted by l_t. In doing so, the operator behaves as a T2 user.

The operator's income per unit demand served is p_t. Our model considers that p_t is decided by the market and not by the operator. As mentioned in [4], such situation arises under perfect competition market model which is widely studied in the economic literature [18].

Our model captures the three-tier hierarchy. The priority of T1 users over T2 users is captured by λ_t. λ_t denotes the number of active channel leases of the operator which gets pre-empted by T1 users at epoch t. The priority of T1 and T2 users over T3 users is captured by M_t^o. M_t^o is the number of channels the operator can use opportunistically for free at epoch t. It is equal to the number of channels which are not being used by T1 and T2 users.

Our system model also penalizes opportunistic use of channels due to their uncertain quality.. This is done using the function $f_t(o_t)$ which is time-varying and is assumed to be *convex* and *monotonically increasing* in o_t. It can have two real world interpretations: *First*, to account for harmful interference in an opportunistic channel, the operator has to transmit at a higher power level[2]. In this case $f_t(o_t)$ respresents the cost to transmit at a higher power level. *Second*, the customers may have lower preference for opportunistic channels compared to leased channels because opportunistic channels may face higher interference. Hence, the operator may charge p_t per unit demand served using leased channels and $\widetilde{p}_t \leq p_t$ per unit demand served using opportunistic channels. This can be captured by setting $f_t(o_t) = (p_t - \widetilde{p}_t)\, o_t$.

B. The Optimization Problem $OP1$

The operator wants to maximize its net profit in optimization horizon T given by

$$\mathcal{P} = \sum_{t=1}^{T} \left(p_t (d_t - g_t) - P l_t \right) = \underbrace{\sum_{t=1}^{T} p_t d_t}_{1^{st}\text{ term}} - \underbrace{\sum_{t=1}^{T} \left(p_t g_t + P l_t \right)}_{2^{nd}\text{ term}} \tag{1}$$

In (1), $p_t(d_t - g_t)$ is the operator's revenue for serving $(d_t - g_t)$ demand at epoch t. P is the price to lease one channel. The operator leases l_t channels at epoch t incuring a net cost of Pl_t. In our model, the operator has no control over d_t and p_t (refer to Section II-A) and hence the 1^{st} term of (1). However it has control over the 2^{nd} term as g_t and l_t are decision variables. Therefore, maximizing $\mathcal{P}$ is equivalent to minimizing the 2^{nd} term. In order to penalize the opportunistic use of channels, we add the function $f_t(o_t)$ to the 2^{nd} term. This leads to the following optimization problem:

$$OP1 \begin{cases} \min\limits_{\{g_t, o_t, l_t\}} \mathcal{C} = \sum\limits_{t=1}^{T} \left(p_t g_t + f_t(o_t) + P l_t \right) \\ \text{subject to: } g_t + o_t + \mathcal{H}(A_t - \lambda_t) \geq d_t \\ \qquad 0 \leq g_t\, ;\; 0 \leq o_t \leq \mathcal{H} M_t^o\, ; \\ \qquad 0 \leq l_t \leq M_t^l - v_t \end{cases}$$

In the first constraint of $OP1$, $A_t = \sum\limits_{i=(t-\tau+1)}^{t} l_i$ is the number of active channel leases at epoch t where τ is the contract duration of a channel lease. However $\lambda_t \leq A_t$ active leases are pre-empted by T1 users[3] leaving effectively $(A_t - \lambda_t)$ active channel leases. One channel can be used to serve $\mathcal{H}$ units of customer demand, where $\mathcal{H}$ is the spectral efficiency. Therefore, $(A_t - \lambda_t)$ channels can be used to serve $\mathcal{H}(A_t - \lambda_t)$ demand. Remaining demand is either rejected, g_t, or served by using opportunistic channels, o_t.

The amount of demand that can be served using opportunistic channels, o_t, cannot exceed $\mathcal{H}M_t^o$ (third constraint).

[2]Transmit power of T3 users cannot cross a threshold as specified by the FCC rules governing 3-TSF.

[3]The SAS will try to relocate the channel of T2 user if it gets preempted by T1 user. λ_t models such relocations of channels too.

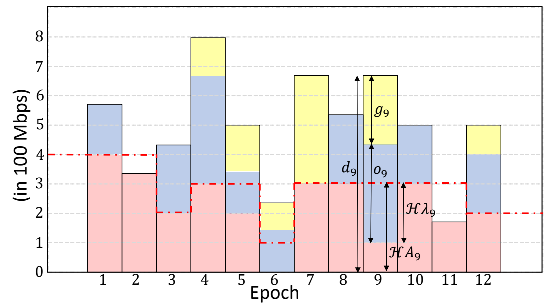

Figure 2. A figure illustrating a typical sequence of events for an operator in 3-TSF. The black histogram represents customer demand, d_t. The dashed red line represents $\mathcal{H}A_t$, the amount of customer demand which can be served using active channel leases. The pink, blue and the yellow regions represent the amount of demand served by active channel leases, $\min(d_t\, ,\, \mathcal{H}(A_t - \lambda_t))$, the amount of demand served by using opportunistic channels, o_t, and the amount of demand rejected, g_t, respectively.

However the operator may choose not to utilize the entire channel capacity $\mathcal{H}M_t^o$ because opportunistic channel use is penalized by function $f_t(o_t)$.

The number of channels leased by the operator, l_t, and the total number of channels leased by all the *other* operators, v_t, at epoch t is decided by the auction conducted in the t^{th} epoch. l_t and v_t must satisfy $l_t + v_t \leq M_t^l$ (fourth constraint), where M_t^l is the number of channels available for leasing at epoch t. The time evolution of M_t^l is governed by[4]

$$M_{t+1}^l = M_t^l - W_t + W_{t-\tau+1} \quad \text{where} \tag{2}$$

$$W_t = l_t + v_t \tag{3}$$

In (2), W_t and $W_{t-\tau+1}$ are the total number of channels that all the operators leases at epoch t and $t-\tau+1$ respectively. Since the contract duration of a lease is τ, $W_{t-\tau+1}$ channel leases re-appear in the market at epoch $(t+1)$.

The input to $OP1$ is the tuple $\varphi_t = (d_t\, ,\, \lambda_t\, ,\, p_t\, ,\, M_t^o\, ,\, v_t\, ,\, f_t(\cdot))$ which consists of six time sequences. The sequence of decision variables g_t, o_t and l_t forms the output of $OP1$. The cost $\mathcal{C}$ incurred by $OP1$ is a function of the sequence φ_t. In $OP1$, all the variables except p_t and P are discrete variables. In particular, the variables $l_t\, ,\, \lambda_t\, ,\, M_t^o\, ,\, M_t^l \in \{0, \ldots, M\}$. The variables $d_t\, ,\, g_t\, ,\, o_t \in \{0, \ldots, d_M\}$; d_M being the maximum demand. The variables d_t, g_t and o_t are expressed in *unit demand*.

Remark 1 (Epoch duration): To enforce three tier hierarchy, the epoch duration should be chosen such that the probability that λ_t or M_t^o change within an epoch is sufficiently low. Optimization of epoch duration has been addressed in [19].

We now consider an example to better understand how the operator serves customer demand in 3-TSF. Figure 2 illustrates a typical sequence of events for the operator. In this example, a channel can serve upto 100 Mbps of customer demand and hence $\mathcal{H} = 100$ Mbps. The contract duration of a lease is $\tau = 8$ epochs. Therefore, a channel leased at epoch 4 expires at epoch 12. Observe that the operator also leases 2 channels at epoch 7. In some epochs all the demand is served using leased channels (like epoch 2). In other epochs some of the demand is served using opportunistic channels (like epoch 3) or it may be rejected (like epoch 7). Of course, the operator can

[4]Equations (2) and (3) is valid even for $t < 1$. However $l_t = 0\, ;\, \forall t < 1$.

combine all the three actions in some epochs (like epoch 5). It is possible that in some epochs (like epoch 6) the operator is not able to use all its active leases because some of them got preempted by T1 users. Epoch 9 illustrates all the key parameters of our system model and how they relate to $OP1$.

C. Assumptions

We first discuss the key assumptions in our system model with justifications.

- A1: We assume that the cost of leasing a channel is a constant P. In practical situations the cost may be time-varying sequence P_t. If P_t belongs to a probability distribution $\mathcal{F}$, then the constant P can be justified as the mean of $\mathcal{F}$. Similar explanation applies to spectral efficiency $\mathcal{H}$ which is expected to be time varying depending on the channel conditions. Without this assumption, the competitive ratio will be unbounded [20]. However, in Section IV, we evaluate the performance of our algorithm using time varying P_t and $\mathcal{H}_t$.
- A2: λ_t and M_t^o are set in the beginning of every epoch and do not change in the entire duration of the epoch. This can be guaranteed by choosing epoch duration appropriately (see Remark 1).

Other than the assumptions on system model, we also need to impose the following assumptions in order to design online algorithms with provable theoretical bounds:

- A3: Moving average of M_t^l over τ epochs is lower bounded by μ_l. Mathematically,

$$0 < \mu_l \leq \frac{\sum_{t=t_o-\tau+1}^{t_o} M_t^l}{\tau} \; ; \; \forall t_o \tag{4}$$

  Qualitatively, μ_l is a measure of the channel availability for leasing. Higher μ_l implies more availability and hence lower wait time. This assumption is used in Proposition 1 to upper bound the wait time. It should be noted that this assumption is not restrictive for the following reasons. *First,* our proposed algorithm in Section III-C does not rely on the knowledge of μ_l. We have used this assumption to derive the competitive ratio of the proposed algorithm in Theorem 4. *Second,* there are many works in literature which assume that the involved random process has a certain mean [21]. Some even assume the entire probability distribution [4], [22]. This assumption is similar with the only difference that unlike these works, we are dealing with time-average instead of ensemble average. *Third,* this assumption can be viewed as a constraint to limit the power of the adversary. Works like [4], [23] dealing with competitive analysis have made similar assumptions.
- A4: $\mathcal{H}p_t$ is upper bounded by p_M, i.e. $\mathcal{H}p_t \leq p_M \, ; \forall t$. The term $\mathcal{H}p_t$ is the maximum revenue which the operator can earn per channel at epoch t. Knowledge of p_M is assumed while designing our algorithm in Section III-C.
- A5: The functions $f_t(x)$ can be evaluated for any x. Evaluation of $f_t(x)$ is the most computationally demanding operation in our algorithm as it may sometimes involve solving an optimization problem [24].

D. Decoupling of $OP1$

We prove here that $OP1$ can be decoupled into two sub-problems. The first sub-problem is to decide the maximum amount of demand to serve by using channels opportunistically. The second sub-problem captures the online nature of leasing channels.

Theorem 1: Let

$$OP2 \begin{cases} \overline{o}_t = \underset{0 \leq o_t \leq \min(d_t \, , \, \mathcal{H}M_t^o)}{\arg\min} \; - p_t o_t + f_t(o_t) \end{cases}$$

Define the following

$$F_t(r_t) \triangleq f_t(\min(r_t \, , \overline{o}_t)) + p_t(r_t - \overline{o}_t)^+ \tag{5}$$

$$D_t \triangleq d_t + \mathcal{H}\lambda_t \tag{6}$$

where D_t is the *effective demand* and $F_t(r_t)$ is the *renting function*. Then the optimal solution g_t^*, o_t^* and l_t^* of $OP1$ can be obtained by solving the optimization problem

$$OP3 \begin{cases} \min\limits_{\{r_t, l_t\}} \; \mathcal{C} = \sum_{t=1}^{T} [F_t(r_t) + Pl_t] \\ \text{subject to: } r_t + \mathcal{H}A_t \geq D_t \\ \qquad 0 \leq r_t \; ; \; 0 \leq l_t \leq M_t^l - v_t \end{cases}$$

for the optimal solution $\overline{r}_t$ and $\overline{l}_t$ and then setting

$$g_t^* = (\overline{r}_t - \overline{o}_t)^+ \; ; \quad o_t^* = \min(\overline{r}_t \, , \overline{o}_t) \; ; \quad l_t^* = \overline{l}_t \tag{7}$$

Proof: Please refer to Appendix A of the supplementary material.

Theorem 1 decouples $OP1$ into $OP2$ and $OP3$. While $OP1$ has three decision variables, $OP2$ has one decision variable and $OP3$ has two decision variables. The inputs to $OP2$ are d_t, p_t, M_t^o and $f_t(\cdot)$. The output of $OP2$ is $\overline{o}_t$, the maximum amount of demand that can be served using opportunistic channels for optimal results.

To get an intuitive understanding of $\overline{o}_t$, let us ignore that o_t is an integer and also satisfies $o_t \leq \mathcal{H}M_t^o$. At $o_t = \overline{o}_t$, the slope of $f_t(o_t)$ is equal to p_t, i.e. $f_t^{'}(\overline{o}_t) = p_t$. This is depicted in Figure 3, where the black curve in Region 2, which is tangent to $f_t(o_t)$ at $o_t = \overline{o}_t$, is parallel to the blue curve. For $o_t < \overline{o}_t$ (Region 1 of Figure 3), $f_t(o_t + 1) - f_t(o_t) < p_t$, implying that the loss incurred by serving a demand using opportunistic channel is less than the loss incurred by rejecting the demand. The opposite is true for $o_t > \overline{o}_t$ (Region 2 of Figure 3). Therefore, the operator will not serve more than $\overline{o}_t$ demand by using channels opportunistically. To solve $OP2$, the operator needs the knowledge of d_t, p_t, M_t^o and $f_t(\cdot)$ for the current epoch. In other words, $OP2$ can be solved using only online information. Also, as discussed in Appendix A, the function $h_t(o_t) = -p_t o_t + f_t(o_t)$ is *unimodal*. We can therefore use tools like *binary search* or *fibonnaci search* [25] to solve $OP2$ in $\mathcal{O}(\log_2(d_M))$ time.

The input to $OP3$ is the tuple $\psi_t = (D_t \, , v_t \, , F_t(\cdot))$ which consists of three time sequences. The sequence of decision variables l_t and r_t forms the output of $OP3$. The variable l_t, as usual, implies leasing channels *(T2)*. The new variable r_t implies *renting*. Mathematically, $r_t = g_t + o_t$. If at epoch t, the operator rejects g_t demand and serves o_t accepted demand using opportunistic channels *(T3)*, then we say that

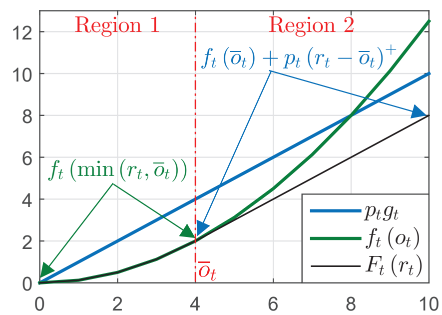

Figure 3. A toy example of a renting function $F_t(r_t)$. In this example $p_t = 1$ and $f_t(o_t) = \frac{1}{8}o_t^2$. By solving $OP2$, we get $\bar{o}_t = 4$. In the above plot, $p_t g_t$ is plotted against g_t, $f_t(o_t)$ is plotted against o_t and $F_t(r_t)$ is plotted against r_t. This figure depicts $\bar{o}_t$ and the two terms of the renting function, $f_t(\min(r_t, \bar{o}_t))$ and $p_t(r_t - \bar{o}_t)^+$, as given by (5).

the operator served $r_t = g_t + o_t$ demand by renting. So in every epoch, the operator has to decide *how much to rent* and *how much to lease* in order to serve D_t effective demand as given by (6). Consider the scenario where the operator serves r_t demand by renting. Out of the r_t demand served by renting, $\min(r_t, \bar{o}_t)$ demand was served by using channels opportunistically incurring a loss of $f_t(\min(r_t, \bar{o}_t))$ while the remaining $(r_t - \bar{o}_t)^+$ demand was rejected incurring a loss of $p_t(r_t - \bar{o}_t)^+$. This is done in order to minimize losses (refer to the previous paragraph). Therefore, the net loss incurred to serve r_t demand by renting is $F_t(r_t)$ as given by (5). $F_t(r_t)$ is called the *renting function* and has the following properties:

Property 1: $F_t(r_t)$ is monotonically increasing in r_t.

Property 1 suggests that for a lease sequence, l_t, and the corresponding sequence of the number of active leases, A_t, the optimal sequence $\bar{r}_t$ which minimizes $OP3$ is given by

$$\bar{r}_t = (D_t - \mathcal{H}A_t)^+ \tag{8}$$

Property 2: First derivative of $F_t(r_t)$ is bounded as follows:

$$F_t(r_t + 1) - F_t(r_t) \le p_t \le \frac{p_M}{\mathcal{H}} \; ; \forall r_t \tag{9}$$

According to Property 2, the operator's loss in an epoch to serve $\mathcal{H}$ demand by renting is atmost p_M. The operator can also serve $\mathcal{H}$ demand by leasing a channel which costs P. Under any practical situation, $p_M \ll P$ implying that serving demand by renting is profitable in the short run. However, the loss incurred to serve $\mathcal{H}$ demand by renting over a period of τ epochs can be τp_M which in general is greater than P, i.e. $\tau p_M > P$. This suggests that leasing is profitable in the long run. This is similar to SRP where renting skis is better in the short run while buying skis is better in the long run. For $OP3$, leasing channels is similar to buying skis while rejecting demand and serving the accepted demand using opportunistic channels is similar to renting skis. This discussion shows why r_t is called renting[5] in order to map $OP3$ with SRP.

Property 3: $F_t(r_t)$ is convex in r_t.

The proof of these properties are straightforward. However, they are included in Appendix B of the supplementary material for the sake of completeness.

[5]"Renting" and "leasing" are indeed synonyms but in this paper they are differentiated based on cost and contract duration. Renting has a contract duration of 1 epoch and cost much less compared to leasing.

$OP3$ captures the online nature of leasing channels. This can be explained as follows. Let $a_i = \sum_{j=i-\tau+1}^{t-1} l_j$ denote the number of active leases in epoch $i \ge t$ if the operator leases zero channels in epoch t. The net rental cost saved by leasing $l_t > 0$ channels in epoch t is

$$\Delta = \sum_{i=t}^{t+\tau-1} \left[F_i\left((D_i - \mathcal{H}a_i)^+\right) - F_i\left((D_i - \mathcal{H}(a_i + l_t))^+\right) \right]$$

A *necessary* condition for the optimality of l_t is $\Delta \ge Pl_t$, i.e. the net rental cost saved by leasing l_t channels should be greater than the cost of leasing l_t channels. To compute Δ, the operator must know D_i, $F_i(\cdot)$; $\forall i \in \{t, \ldots, t+\tau-1\}$. To calculate D_i, $F_i(\cdot)$ for $i > t$, the operator needs future knowledge of ψ_i (or equivalently φ_i) . This suggest that online information is not enough to decide the optimal l_t.

Remark 2 (Optimal algorithm for $OP3$): Optimal algorithm for $OP3$ needs offline information, i.e. the entire sequence ψ_t ; $\forall 1 \le t \le T$ should be known in advance. The optimal algorithm can be formulated as a dynamic programming problem similar to [14, Section 3]. It has a pseudo-polynomial time complexity of $\mathcal{O}(T(M^\tau + \log_2(d_M)))$ which is intractable under any practical scenario. A detailed discussion of the optimal algorithm is not required to understand the online algorithm for $OP3$. However, we have included it in Appendix C of the supplementary material for the sake of completeness.

Remark 3 (Comparison with [14] *and theoretical contribution):* $OP3$ resembles the optimization problem considered in [14]. In [14], a cloud computing user has to decide the number of virtual machines it wants to reserve (similar to leasing) and the number of virtual machines it wants to use on-demand (similar to renting). A cloud computing user may reserve as many virtual machines it wants but the operator cannot lease more than $M_t^l - v_t$ channels at t^{th} epoch. This is the major difference between our work and [14]. Up to our knowledge, no work in ski-rental literature has dealt with similar situations. Designing and analysing online algorithms for such situations constitutes the theoretical contribution of this paper.

III. Deterministic Online Algorithm

This section contains the main result of the paper, a deterministic online algorithm for leasing channels. We approach this in steps. In Section III-A we propose a special case of $OP3$ called the *Modified Ski-Rental Problem* (MSRP) and show that is not possible to get a practically viable competitive ratio for MSRP without constraining M_t^l. Having proposed MSRP, we design an optimal deterministic online algorithm to solve MSRP in Section III-B. Study of MSRP leads to two outcomes. *First*, it suggests a possible structure of the online algorithm for $OP3$. *Second*, it provides a lower bound on the competitive ratio which no online algorithm for $OP3$ can break. Using the insights drawn from studying MSRP, we design and analyze a deterministic online algorithm for leasing channels in Section III-C.

Competitive ratio preliminaries: The operator has to decide (r_t, l_t) using only the knowledge of ψ_i till the t^{th} epoch. This has to be done in a certain optimal sense called the competitive ratio (CR). CR is a relative measure of an online algorithm with respect to an optimal algorithm. Define the sequence $\psi = \{\psi_1, \psi_2 \ldots, \psi_T\}$. Let $\mathcal{C}_{\mathcal{A}}(\psi)$ and $\mathcal{C}_{opt}(\psi)$ be the cost incurred by a deterministic online algorithm $\mathcal{A}$ and the optimal algorithm *opt* respectively. $\mathcal{A}$ is $c-$competitive iff

$$c = \sup_{\psi \in \mathcal{S}} \frac{\mathcal{C}_{\mathcal{A}}(\psi)}{\mathcal{C}_{opt}(\psi)}$$

where the set $\mathcal{S}$ contains all possible values of ψ. A smaller c implies a better online algorithm. Competitive analysis is often thought of as a two player game between an *adversary* which generates ψ to maximize the ratio $\frac{\mathcal{C}_{\mathcal{A}}(\psi)}{\mathcal{C}_{opt}(\psi)}$ and the online algorithm $\mathcal{A}$ which tries to minimize the ratio.

A. Modified Ski-Rental Problem

In this section we propose a modification of the classical SRP called MSRP as follows:

1) A skier needs one ski[6] a day. Skiing vacation is at most τ days (equal to the lease period) but can end on the y^{th} day (where $0 \le y \le \tau$) if the skier gets injured while skiing. In the context of $OP3$, the effective demand structure is : $D_t = 1\,;\, 1 \le t \le y$ and $D_t = 0\,;\, t > y$.
2) A shop *rents* out a ski for p_M dollars per day and *leases* out a ski for P dollars, where $p_M \ll P$. The lease period is $\tau > 1$ days. In the context of $OP3$, $F_t(r_t) = p_M r_t$.
3) A ski can serve only one skier at a time. In the context of $OP3$, $\mathcal{H} = 1$.
4) The shop has a total of M skis for lease. The number of skis available for leasing on the t^{th} day is M_t^l where M_t^l is governed by (2) and (3). For MSRP, l_t and v_t are the number of skis "the skier" and the "other skiers" lease on day t respectively.
5) Skis are available for leasing on the first day. In the context of $OP3$, $M_1^l > 0$.

The above five points shows that $OP3$ can be reduced to MSRP by constraining D_t, $F_t(r_t)$, $\mathcal{H}$ and M_1^l. Hence, MSRP is a special case of $OP3$. If the shop has infinitely many skis to lease; $M = \infty$, then there will always be skis available for leasing; $M_t^l > 0\,;\, \forall t$. In this case MSRP reduces to SRP. For SRP, the well known optimal online deterministic algorithm is the *breakeven algorithm* which can be stated as follows. Say the skier is still skiing on the k^{th} day. If the net renting cost $p_M k \ge P$, the skier should lease a ski on the k^{th} day. Else, the skier should rent. CR of this algorithm is 2.

If M is finite then it is possible that $M_t^l = 0$ for some t. The key difference between SRP and MSRP is the *availability of ski leases*. The skier may decide to lease on the k^{th} day only to find that $M_k^l = 0$ because the other skiers have leased all the M skis. Without any constraint on M_t^l, the wait time of the skier to purchase a ski may be infinite. In worst case scenario, the skier has to keep renting till her vacation ends incurring a cost of τp_M while the offline algorithm which can foresee the future will lease a ski on the 1^{st} day. Hence the CR is $\frac{\tau p_M}{P}$. This discussion leads to the following theorem.

Theorem 2: In the absence of any constraint on M_t^l, no online algorithm for MSRP can achieve a CR lesser than $\frac{\tau p_M}{P}$.

[6]From now on, "a ski" or "one ski" implicitly means a pair of skis.

Theorem 2 extends to $OP3$ as well because MSRP is a special case of $OP3$. We therefore constrain M_t^l using (4). In (4), μ_l characterizes the average availability of channel/ski leases in the market. Higher the availability, lower the wait time. In the following, we give a formal definition of waitime and upper bound it using assumption A3.

Definition 1 (Wait Time): Say that the skier/operator decides to purchase l leases at epoch t_l. The wait time η is the minimum number of epochs the skier/operator has to wait to purchase all the l leases. Mathematically,

$$\eta = \inf\left\{\delta \ge 0 \,|\, \sum_{t=t_l}^{t_l+\delta} M_t^l \ge l + \sum_{t=t_l}^{t_l+\delta} v_t\right\} \quad (10)$$

In (10), $l + \sum_{t=t_l}^{t_l+\delta} v_t$ is the net demand of lease in the time period $[t_l\,,\, t_l+\delta]$ while $\sum_{t=t_l}^{t_l+\delta} M_t^l$ is the net channel/ski lease sold in the time period $[t_l\,,\, t_l+\delta]$.

Proposition 1: If moving average of M_t^l is lower bounded by μ_l (assumption A3) , then $\eta \le \eta_M(\mu_l)$, where $\eta_M(\mu_l)$ can be characterized as follows. If $\mu_l = \frac{M}{\tau}$, $\eta_M(\mu_l) = \infty$. For $\mu_l > \frac{M}{\tau}$, consider the following linear inequalities in $M^l = \left\{M_0^l, M_1^l, \ldots, M_{\tau-1}^l\right\} \in \mathbb{Z}_+^\tau$

$$M_0^l = M \quad (11)$$

$$M_{t+1}^l \le M_t^l\,;\, 0 \le t \le \tau-2 \quad (12)$$

$$\sum_{t=\theta+1}^{\tau-1} M_t^l - \sum_{t=\eta}^{\theta-1} M_t^l \ge \tau\mu_l + (\theta-\eta)\,w + (\theta-\eta)\,M_{\tau-1}^l - (\theta-\eta+1)\,M\,;\, \eta \le \theta \le \tau-1 \quad (13)$$

$$M_\eta^l - M_{\tau-1}^l \le M - w \quad (14)$$

$$-M_{\eta-1}^l + M_{\tau-1}^l \le -M + w - 1 \quad \text{then,} \quad (15)$$

$$\eta_M(\mu_l) = \sup\{0 \le \eta \le \tau-1 \mid 0 \le w \le M\,;\, \text{(11)-(15) are simultaneously feasible in } M^l \text{ if } \eta > 0 \text{ or (11)-(14) are simultaneously feasible in } M^l \text{ if } \eta = 0\} \quad (16)$$

Proof: Please refer to Appendix D of the supplementary material.

In (13), θ is an index variable and in (13)-(15), w is the number of leases the skier/operator *decides* to purchase at a given epoch. For any μ_l satisfying $\frac{M}{\tau} < \mu_l \le M$, the time sequence $M_t^l = M\,;\, 0 \le t \le \tau-1$ satisfies (11)-(14) if $\eta = 0$ and $w = 0$. Hence, there exists an $\eta_M(\mu_l)$ for any μ_l satisfying $\frac{M}{\tau} < \mu_l \le M$. Proposition 1 gives a mechanism to find $\eta_M(\mu_l)$, the maximum wait time η_M for a given μ_l. This can be done by starting from $\eta = \tau$ and then decreasing η till (11)-(15) are simultaneously feasible in M^l. Inequalities (11)-(15) constitutes a Integer Program which can be solved using solvers like Gurobi[7]. A typical plot of $\eta_M(\mu_l)$ is shown later in Section III-C. We would like to stress that the workings of Algorithm 1 and Algorithm 2 do not rely on the computation of $\eta_M(\mu_l)$. $\eta_M(\mu_l)$ is calculated only to find the CR of these algorithms. Hence, time complexity of the integer program does not effect the time complexity of these algorithms.

[7]If M is large, we can approximate the integer program with a linear program (continous) to improve the time complexity.

B. Online Algorithm for MSRP

In this section, we design an online algorithm for MSRP which has the best CR, assuming that the wait time $\eta \leq \eta_M(\mu_l)$. This gives us insights into designing a deterministic online algorithm for leasing channels.

The offline algorithm can foresee the y^{th} day when the skier will get injured. It leases a ski on the 1^{st} day if $yp_M \geq P$, else it keeps renting a ski everyday till the y^{th} day. Hence, the cost incurred by the offline algorithm is

$$C_{off} = \min(yp_M, P) \tag{17}$$

The online algorithm does not know y in advance. Our objective is to design an optimal online algorithm which decides if and when to lease a ski. We first consider online algorithms having the structure $n \to b$. This structure can be explained as follows. The skier decides to lease on the n^{th} day if he/she is still skiing. After deciding to lease, the skier waits till the $(n+\eta)^{th}$ day when the leases are available again. On the $(n+\eta)^{th}$ day, the skier may lease a ski if he/she is still skiing ($b = 1$) or keep renting till the vacation ends ($b = 0$). We first study the case when $b = 1$, i.e. the skier definitely leases after the wait time.

In MSRP, the scalar variable y and the sequence v_t are the inputs to the online algorithm. In competitive analysis, an adversary chooses y and v_t to maximize the CR. If the skier leases a ski on the k^{th} day, the adversary will injure the skier on the k^{th} day (i.e. $y = k$), since waiting further can only increase the offline cost C_{off} without increasing the online cost C_{on}. The adversary controls the wait time η by setting the time sequence v_t. This is because v_t decides M_t^l (see (2) and (3)) and hence the wait time η. If the skier decides to lease on the n^{th} day, CR as a function of n is

$$c(n) = \sup_{0 \leq \eta \leq \eta_M} \frac{np_M + \eta p_M + P}{\min(np_M + \eta p_M, P)} \tag{18}$$

In (18), the skier lease a ski on $(n+\eta)^{th}$ day. Hence, the online cost is $C_{on} = (n+\eta)p_M + P$. As discussed before, the adversary will injure the skier on the $(n+\eta)^{th}$ day to maximize CR. Hence, $C_{off} = \min((n+\eta)p_M, P)$. To simplify (18), we consider the following two cases:

Case-1 ($np_M + \eta p_M < P$): In this case, $c(n) = \sup_{0 \leq \eta \leq \eta_M} \frac{np_M + \eta p_M + P}{np_M + \eta p_M}$. Consider the inequality $\frac{x+A}{x+B} \leq \frac{A}{B}$ which holds if $x \geq 0$ and $A \geq B > 0$. In $\frac{np_M + \eta p_M + P}{np_M + \eta p_M}$, $x = \eta p_M$, $A = np_M + P$ and $B = np_M$. Hence, $c(n) = \sup_{0 \leq \eta \leq \eta_M} \frac{np_M + \eta p_M + P}{np_M + \eta p_M} = \frac{np_M + P}{np_M}$.

Case-2 ($np_M + \eta p_M \geq P$): In this case, $c(n) = \sup_{0 \leq \eta \leq \eta_M} \frac{np_M + \eta p_M + P}{P} = \frac{np_M + \eta_M p_M + P}{P}$.

Based on Case-1 and Case-2, (18) can be simplified as

$$c(n) = \max\left(\frac{np_M + P}{np_M}, \frac{np_M + \eta_M p_M + P}{P}\right) \tag{19}$$

The online algorithm should select n to minimize $c(n)$ in (19). In (19), the functions $\frac{np_M + P}{np_M}$ and $\frac{np_M + \eta_M p_M + P}{P}$ are monotonically decreasing and monotonically increasing repsectively for $n > 0$. Hence, the optimal $n = n_{op}$ which minimizes $c(n)$ can be obtained by equating $\frac{np_M + P}{np_M}$ and $\frac{np_M + \eta_M p_M + P}{P}$. This is shown in Figure 4. Equating $\frac{np_M + P}{np_M}$ and $\frac{np_M + \eta_M p_M + P}{P}$ we get,

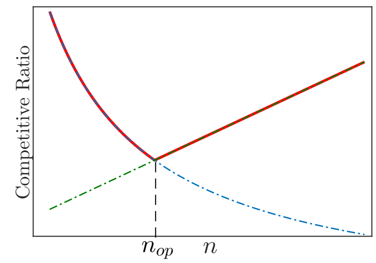

Figure 4. The dashed-blue curve, the dashed-green curve and the solid-red curve are typical plots of functions $c_1(n) = \frac{np_M + P}{np_M}$, $c_2(n) = \frac{np_M + \eta_M p_M + P}{P}$ and $c(n) = \max(c_1(n), c_2(n))$ respectively. Note that the minima of $c(n)$, denoted as n_{op}, is the intersection of $c_1(n)$ and $c_2(n)$.

$$z_{op}^2 + \eta_M p_M z_{op} - P^2 = 0 \tag{20}$$

where $z_{op} = n_{op} p_M$, is the optimal net renting cost after which the skier should decide to lease a ski. For $z = z_{op}$, the CR is

$$c(z_{op}) = \left(1 + \frac{z_{op}}{P}\right) + \frac{\eta_M p_M}{P} \tag{21}$$

Remark 4 (Intuition behind $z_{op} \leq P$): Note that $z_{op} \leq P$. In SRP, a ski is leased when the net rental cost reaches a threshold of P. MSRP on the other hand has a threshold of z_{op} which is less than P. This *proactive* nature of leasing in MSRP arises due to the risk of M_t^l becoming 0 in future epochs.

To this end, we only discussed the case when $b = 1$, i.e. the skier definitely leases after the wait time. However, this is not always the optimal strategy. Consider the following cases:

Case-A ($b = 1$): In this case the skier leases after the wait time of η. If the skier decided to lease after incurring a rental cost z, then CR is $\frac{z + \eta p_M + P}{P}$.

Case-B ($b = 0$): In this case the skier keeps renting till the end of vacation and hence the CR is $\frac{\tau p_M}{P}$.

It is optimal to lease after the wait time only if

$$\frac{z + \eta p_M + P}{P} \leq \frac{\tau p_M}{P} \iff \eta \leq \tau - \frac{(z+P)}{p_M} \tag{22}$$

For $z = z_{op}$, inequality (22) is guaranteed if

$$\eta_M \leq \tau - \frac{(z_{op} + P)}{p_M} \tag{23}$$

Algorithm 1 and Proposition 2 summarize our discussion. Nnote that Algorithm 1 is based on two thresholds. *First*, z_{op} as defined in (20). z_{op} is the optimal net renting cost after which the skier decides to lease a ski. This is implemented in *line 3* of Algorithm 1. *Second*, η_M as defined in (23). η_M is the maximum wait time after which the skier rejects its decision to lease a ski. This is implemented in *line 6* of Algorithm 1.

Proposition 2: Among all online algorithms for MSRP with structure $n \to b$, Algorithm 1 has the best CR of $c_{opt}(\mu_l) = c_{opt}(\eta_M(\mu_l))$ where $\eta_M(\mu_l)$ is given by (16) and

$$c_{opt}(\eta_M) = \begin{cases} \left(1 + \frac{z_{op}}{P}\right) + \frac{\eta_M p_M}{P} & ; \eta_M \leq \tau - \frac{(z_{op}+P)}{p_M} \\ \frac{\tau p_M}{P} & ; \eta_M > \tau - \frac{(z_{op}+P)}{p_M} \end{cases} \tag{24}$$

Algorithm 1: A deterministic online algorithm for MSRP.

```
Initialize decided = 0 and leased = 0. decided = 1 if
the skier decides to lease a ski and 0 otherwise.
leased = 1 if the skier leases a ski and 0 otherwise.
repeat
    if sum_{i=1}^{t} p_i >= z_op AND decided = 0 then
        The skier decides to lease a ski. Hence, set
        decided = 1. Also, set t_l = t indicating the day
        when the skier decided to lease a ski.
    end
    if decided = 1 AND leased = 0 AND M_t^l > 0 AND
    t - t_l <= eta_M then
        The skier leases a ski. Hence, set leased = 1.
    end
    if leased = 0 then
        The skier rents a ski.
    end
until "The skier is injured"
```

Proposition 3: Algorithm 1 achieves the best CR for MSRP.

Proof: The universe of all possible online algorithms for MSRP can be abstracted as follows:

$$\underbrace{n_1 \to b_1}_{1^{st}\text{ Stage}} \to \underbrace{n_2 \to b_2}_{2^{nd}\text{ Stage}} \to \underbrace{n_3 \to b_3}_{3^{rd}\text{ Stage}} \cdots$$

The above abstraction is divided in stages. Each stage has the same structure as that of $n \to b$ discussed before. Transition from stage i to stage $i+1$ happens if in stage i the skier decides not to lease after the wait time (i.e. $b_i = 0$) but then at a later time it decides to lease again.

Say that in the 1^{st} stage, the skier uses Algorithm 1 and decides not to lease after the wait time, i.e. $b_1 = 0$. This happens when the wait time $\eta > \tau - \frac{(z_{op}+P)}{p_M}$. The maximum possible renting cost after the wait time is

$$p_M\left(\tau - n_{op} - \eta\right) < p_M\left(\tau - \frac{z_{op}}{p_M} - \tau + \frac{(z_{op}+P)}{p_M}\right) = P$$

Since the maximum renting cost after the wait time is lesser than the cost of a lease, it is better to keep renting till the end of vacation. Hence if the skier decides not to lease in the 1^{st} stage, it is not optimal to lease at a later time. Therefore 2^{nd} stage (and hence the later stages) is not required to design an optimal algorithm; 1^{st} stage is sufficient. Hence, Algorithm 1 achieves the best CR for MSRP.

Theorem 3: An online algorithm for $OP3$ cannot achieve a CR lesser than $c_{opt}(\mu_l)$.

Proof: This directly follows from the fact that MSRP is a special case of $OP3$.

C. Online Algorithm for Leasing Channels

Motivated by the online algorithm for MSRP designed in Section III-A, we suggest a *threshold based* algorithm for leasing channels. There are two threshold criteria:

1) The operator decides to lease a channel when the net incremental renting cost exceeds z_{th}.
2) The operator rejects the decision to lease a channel if the wait time exceeds $\tau - \frac{(z_{th}+P)}{p_M}$.

Algorithm 2: $\mathcal{A}_{z_{th}}$: a deterministic online algorithm for leasing channels in Three-Tier Spectrum Sharing Framework.

```
1  Initialize a time sequence a_t. Set a_t = 0, ∀t. a_t is the
   virtual number of active leases at epoch t.
2  Repeat steps 3-10 for all epochs. Let current epoch be t.
3  Learn d_t, p_t, λ_t, M_t^o and f_t(o_t).
4  Compute ō_t by solving OP2. Set D_t = d_t + Hλ_t.
5  repeat
6      Compute the net incremental rental cost R from epoch
       t - τ + 1 to current epoch t.
7      if R >= z_th then
8          The operator decides to lease a channel. Hence the
           current epoch t is enqueued into the FIFO queue.
9          Set a_i = a_i + 1; i = t-τ+1, ..., t-1 to update the
           history of a_i's. This shows that previous mistakes
           have been accounted.
10         Set a_i = a_i + 1; i = t, ..., t+τ-1 to updates
           future a_i's. This show that an additional virtual
           lease is available in future epochs.
11     end
12 until R >= z_th
13 repeat
14     Read timestamp from the FIFO queue. Let the time
       stamp read t_l. Set wait time η = t - t_l.
15     If η > τ - (z_th+P)/p_M, then Dequeue timestamp from the
       FIFO queue.
16 until η > τ - (z_th+P)/p_M AND "FIFO Queue is Not Empty"
17 Find the number of timestamps in the FIFO queue. Let it
   be L_t. Place a bid for min(L_t, M_t^l) channel leases in the
   current auction.
18 Let the operator win l_t channel leases. Dequeue l_t times-
   tamps from the FIFO queue.
19 Number of active lease is A_t = sum_{i=t-τ+1}^{t} l_i. Serve
   min(D_t, HA_t) demand using active leases.
20 Set r_t = (D_t - HA_t)^+. Serve o_t = min(r_t, ō_t)
   demand by using channels opportunistically. Reject
   g_t = (r_t - ō_t)^+ demand
```

In Algorithm 2, we present a threshold based algorithm $\mathcal{A}_{z_{th}}$ based on a generic threshold z_{th}. However in this paper we only consider $\mathcal{A}_P$, i.e. $\mathcal{A}_{z_{th}}$ with $z_{th} = P$.

Remark 5 (Why $z_{th} = P$?): Our analysis in Section III-B, suggests that $\mathcal{A}_{z_{op}}$ should have the best CR. To find z_{op} we need knowledge of μ_l, which depends on market statistics. Our key motivation in this work is to design an online algorithm for leasing channels that does not rely on the knowledge of market statistics. We therefore explore $\mathcal{A}_P$[8] and compare its CR with $c_{opt}(\mu_l)$, the optimal CR for $OP3$.

Algorithm 2 can be divided into five steps.

[8] $z_{th} = P$ has special significance because in classical SRP, P is the breakeven threshold for leasing a ski.

Step 1 [Line 3 (Learn φ_t)]: Recall that the tuple $\varphi_t = (d_t\,, \lambda_t\,, p_t\,, M_t^o\,, v_t\,, f_t\,(\cdot))$ is the input to $OP1$. At epoch t, the operator knows d_t and p_t. If the SAS preempts λ_t channels at epoch t, then the operator can use only $(A_t - \lambda_t)$ active channel leases. Since the operator knows A_t, it can find λ_t. We assume that M_t^o can be learned by querying the SAS[9]. $f_t\,(\cdot)$ depends on the state of the opportunistic channels which can be estimated, possibly using QoS reviews from the customers. It is to be noted that our algorithm *does not* assume the knowledge of v_t.

Step 2 [Line 4 (Calculate $\overline{o}_t$)]: The operator computes $\overline{o}_t$, the maximum amount of demand that can be served using free opportunistic channels for optimal results *(line 4)*. This involves solving $OP2$ which can be done using binary/ fibonacci search. The renting function $F_t\,(r_t)$ is implicitly dependent on $\overline{o}_t$; see (5). Hence we need to compute $\overline{o}_t$ in order to evaluate $F_t\,(r_t)$ in Step 3. The operator also computes the effective demand D_t using (6).

Step 3 [Line 5-12 (Deciding to Lease or Not)]: The operator maintains a time sequence a_t, the *virtual* number of active leases at epoch t. The reason why a_t is *virtual* will be made obvious shortly. The sequence a_t helps the operator to decide the number of channels it wants to lease in the current epoch. At current epoch t, the operator looks back τ epochs and calculates the net incremental renting cost *(line 6)*. Net incremental renting cost $\mathcal{R}$ is the net renting cost[10] which could have been saved in the time period $[t-\tau+1, t]$ if the operator has one additional lease. Mathematically, $\mathcal{R} = \sum\limits_{i=t-\tau+1}^{t} \gamma\,(a_i)$ where

$$\gamma\,(a_i) = F_i\left(\left(D_i - \mathcal{H}a_i\right)^+\right) - F_i\left(\left(D_i - \mathcal{H}\,(a_i+1)\right)^+\right) \tag{25}$$

In (25), $r\,(a_i)$ is the incremental renting cost at epoch i. As shown in Figure 5, $r\,(a_i) = 0$ if $\mathcal{H}a_i \geq D_i$. $F_i\left(\left(D_i - \mathcal{H}a_i\right)^+\right)$ and $F_i\left(\left(D_i - \mathcal{H}\,(a_i+1)\right)^+\right)$ are the renting costs in the i^{th} epoch to serve the demand above the red and the blue graphs respectively in Figure 5, respectively.

If $\mathcal{R} \geq P$ then the operator could have minimized the loss by leasing a channel in epoch $t-\tau+1$. To compensate for this mistake the operator decides to lease a channel. The current timestamp t is *enqueued* in the FIFO queue *(line 8)* as shown in Figure 6. A virtual lease is purchased at epoch $t-\tau+1$ to indicate that a corrective measure has been taken for the past mistake *(line 9)*. Without such update the operator will take corrective measure for the same mistake multiple times. The future a_i's are also updated assuming that the operator can purchase an additional lease in the current epoch *(line 10)*. The discussion in this paragraph shows that the operator may not have a_t active lease at epoch t and hence it is called the *virtual* number of active leases.

[9]The rules governing what SAS can reveal to the operators are still under consideration by the FCC. Hence, it is not clear if the SAS can reveal M_t^o to the operators. However, if the SAS can reveal M_t^o to the operators, it is likely to do so *truthfully*. We assume that this is because SAS is a federal entity.

[10]The net incremental renting cost $\mathcal{R}$ should not be confused with the net renting cost Δ defined in Page 6. While $\mathcal{R}$ is calculated looking backward in time, Δ is calculated looking forward in time. Also, $\mathcal{R}$ is the net renting cost which could have been saved by leasing one additional channel. However, the definition of Δ considers that the operator can lease more than one channel.

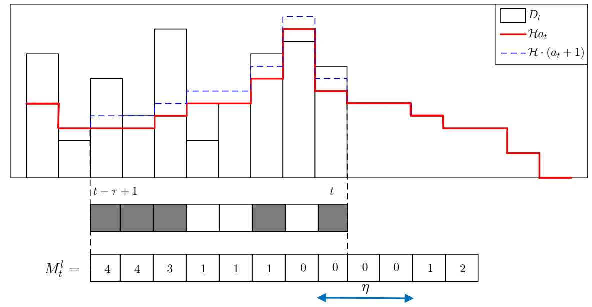

Figure 5. An illustration of net incremental renting cost and wait time. When $\mathcal{H}a_t < D_t$ ($\mathcal{H}a_t \geq D_t$), i.e. the red graph is below (above) the black graph, a non-zero (zero) incremental renting cost $r\,(a_t)$ is incurred. This is depicted using grey (white) epochs in the upper strip. In this example, wait time $\eta = 3$.

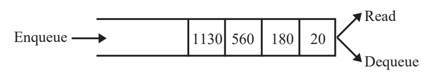

Figure 6. A FIFO Queue containing time stamps. Timestamps are enqueued behind the queue while they are read and dequeued from the front of the queue.

Step 3 is repeated until $\mathcal{R} < P$. When $\mathcal{R} < P$, it indicates that purchasing any more additional lease is costlier than renting. This is a direct consequence of Property 3. Hence the operator decides not to purchase any additional leases.

Step 4 [Line 13-18 (To Lease or Not)]: The operator starts by *dequeueing* the timestamps from the FIFO queue whose corresponding wait time $\eta > \tau - \frac{2P}{p_M}$ *(line 13-16.)*. Let L_t denote the number of time stamps in the FIFO queue after this operation. The wait time corresponding to these L_t timestamps is less than $\tau - \frac{2P}{p_M}$ and hence the operator wants to lease L_t channels in the current epoch. But there may be only M_t^l channels available for leasing. Hence the operator places a bid for $\min\left(L_t\,, M_t^l\right)$ channels *(line 17)*. If the operator wins $l_t \leq \min\left(L_t\,, M_t^l\right)$ channels, it *dequeues* l_t timestamps from the FIFO queue *(line 18)* indicating that a channel has been leased corresponding to each of these l_t leasing decisions.

A FIFO queue is used to process timestamps in the order in which they were generated. Otherwise it is possible that the wait time of a timestamp, which could have been below the threshold $\tau - \frac{2P}{p_M}$ gets rejected because it was processed at a later epoch.

Step 5 [Line 19-20 (Calculate o_t and g_t)]: If there are A_t active leases, then by Property 1, $r_t = (D_t - \mathcal{H}A_t)^+$ demand are served by renting *(line 20)*. Finally the amount of demand to serve using opportunistic channels and the amount of demand to reject is given by (7).

Theorem 4: If the moving average of M_t^l is lower bounded by μ_l (assumption A3), then the CR of $\mathcal{A}_P$ is $c_P\,(\mu_l) = c_P\,(\eta_M\,(\mu_l))$ where $\eta_M\,(\mu_l)$ is given by (16) and

$$c_P\,(\eta_M) = \begin{cases} 2 + \frac{\eta_M p_M}{P} & ;\, \eta_M \leq \tau - \frac{2P}{p_M} \\ \frac{\tau p_M}{P} & ;\, \eta_M > \tau - \frac{2P}{p_M} \end{cases} \tag{26}$$

Proof: Please refer to Appendix E of the supplementary material.

In Figure 7 we show a typical plot of maximum wait time η_M and CR of $\mathcal{A}_P$ as a function of μ_l. Figure 7.b. shows that the CR of $\mathcal{A}_P$ is close to the optimal CR $c_{opt}\,(\mu_l)$.

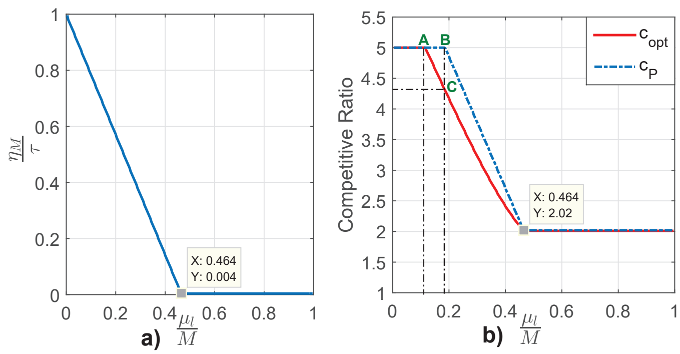

Figure 7. (a) A typical plot of normalized maximum wait time η_M, i.e. $\frac{\eta_M}{\tau}$, vs normalized μ_l, i.e. $\frac{\mu_l}{M}$. It is obtained by solving the IP Feasibility Problem stated in Proposition 1 for various values of μ_l. In a typical plot, $\eta_M(\mu_l) = 0\,; \forall \mu_l \geq M/2$. This plot does not show the isolated case when $\eta_M(\mu_l) = \infty$ for $\mu_l = \frac{M}{\tau}$. (b) A typical plot comparing the CR of $\mathcal{A}_P$ (Theorem 4) with the most optimal CR (Proposition 2). It should be noted that when $\mu_l = \frac{M}{\tau}$, maximum wait time $\eta_M(\mu_l) = \infty$, however $c_P(\mu_l) = c_{opt}(\mu_l) = \frac{\tau p_M}{P}$, a finite CR.

Theorem 5: The time complexity of $\mathcal{A}_P$ is $\mathcal{O}(\log_2(d_M) + 2\tau M)$.

Proof: By assumption A5, the evaluation of $f_t(\cdot)$ contributes to the time complexity of $\mathcal{A}_P$. All other operations in an epoch is absorbed (up to a constant factor) by the time taken for evaluating $f_t(\cdot)$. To compute $\overline{o}_t$ (*line* 4), $f_t(\cdot)$ has to be evaluated $\mathcal{O}(\log_2(d_M))$ times (provided that we are using binary/ fibonacci search). To compute $\mathcal{R}$ in every iteration of the repeat-until loop (*line 6*), $f_t(\cdot)$ has to be evaluated $\mathcal{O}(2\tau)$ times. There is a maximum of M channels to lease and hence there can be at most M iterations of the while loop. Hence there are $\mathcal{O}(\log_2(d_M) + 2\tau M)$ evaluations of $f_t(\cdot)$ in a given epoch.

IV. Simulation Results

In this section we present simulations that compare the online algorithm $\mathcal{A}_P$ with a number of benchmark algorithms. We also study how $\mathcal{A}_P$ utilizes opportunistic channels as a function of a few trace parameters. Since real-world traces of mobile operators are not available in public domain, we use synthetic traces in our simulations. In doing so we can evaluate the performance of $\mathcal{A}_P$ under various statistical properties of the traces.

Setup and trace generation: We start by defining the function $f_t(x)$ which penalizes opportunistic channel use. In our simulation, $f_t(x)$ captures the power (and hence the cost) required to serve demand using opportunistic channels. Channels are assumed to have Shannon capacity. Hence $f_t(x) = N_t\left(2^{\frac{x}{M_t^o}} - 1\right)$ where the channel bandwidth has been normalized to 1 and N_t is the average noise power experienced by opportunistic users at epoch t. For the chosen $f_t(x)$, the solution for the unconstrained $OP2$ is $\overline{o}_t = \frac{M_t^o}{\log(2)} \log\left(\frac{M_t^o p_t}{N_t \log(2)}\right)$. We would like to control $\overline{o}_t$ such that $\overline{o}_t = \beta_t \mathcal{H} M_t^o$ where $\beta_t \in (0, 1]$ is the *quality factor* which governs the fraction of the available opportunistic channel capacity, $\mathcal{H} M_t^o$, which should be used for optimal performance. This can be done by setting $N_t = \frac{M_t^o p_t}{2^{\mathcal{H}\beta_t} \log(2)}$.

Time sequences d_t, λ_t, p_t, M_t^o, v_t and the quality factor β_t forms the input to $OP1$. We model these time sequences as discrete time markov chain (DTMC). This is motivated by the existing literature: for d_t, p_t, v_t (see [4]), for λ_t, M_t^o (see [26]) and for β_t (see [27]). The mean and coefficient of variation (CV[11]) of the stationary distribution of all the six DTMC's can be controlled[12].

Table II
Default Simulation Settings.
1 epoch $= 1$ hour, $\tau = 1$ year, $T = 10\tau$, $p_M = 1$, $\frac{\tau p_M}{P} = 5$, $M = 50$, $d_M = 15$, $\mathcal{H} = 1$, Winning probability of the operator$= 0.5$

Trace	# of States	State Space	Mean (μ)	CV ($\frac{\sigma}{\mu}$)
d_t	$d_M + 1$	$\{0, \ldots, d_M\}$	4	0.9
M_t^o	$M + 1$	$\{0, \ldots, M\}$	2	0.5
p_t	50	$\{0.8p_M, \ldots, p_M\}$	$0.95p_M$	0.05
β_t	50	$\{0.05, \ldots, 1\}$	0.66	0.35
λ_t	$M + 1$	$\{0, \ldots, M\}$	$\frac{M}{10}$	1
v_t	3	$\{0, \ldots, 2\}$	$\frac{M}{\tau}$	0.05

Default simulation settings are shown in Table II. These settings are used in the simulations unless stated otherwise. We assume that the operator wins a channel with probability 0.5. Based on [2], [3], we choose an epoch duration of 1 hour, $\tau = 1$ year, $M = 50$ channels. p_M is normalized to 1 and P is set such that $\frac{\tau p_M}{P} = 5$. Default trace properties of the six time sequences are set to some acceptable value as tabulated in Table II. Two of these trace properties needs further explanation. First, we assumed that the channel occupancy of T1 users is 10%. Hence we choose mean of λ_t as $\frac{M}{10}$. Second, a mean v_t of $\frac{M}{\tau}$ implies that it takes an average of τ epochs for the other operators to purchase all the M channel leases. The default value of spectral efficiency $\mathcal{H}$ is 1. This is because of simulation constraints. If we choose a higher spectral efficiency, we have to simultaneously increase the maximum demand d_M to conduct any meaningful simulations. The following issues are encountered as d_M increases: (a) Time complexity of the MDP algorithm (described later in this section) increases. (b) Difficulty in convergence of the optimal algorithm. (c) Higher RAM requirement to store the markov matrix of d_t. Hence $\mathcal{H}$ is set to 1 for most of the simulations except when we study the effect of varying spectral efficiency. We would like to stress that the simulation results will not change even if $\mathcal{H}$ is large. This is because our simulation results study normalized cost, the ratio of two costs. If $\mathcal{H}$ increases, both the costs will increase proportionally. Hence, the normalized cost will remain the same.

In the first half of this section, we compare $\mathcal{A}_P$ with some benchmark algorithms. To do this we use the following definition of *normalized cost*: "Cost incurred by $\mathcal{A}_P$ to the cost incurred by the benchmark algorithm."

Comparison with trivial online algorithms: We compare $\mathcal{A}_P$ with two trivial online algorithms: *i) Opportunistic use only:* This algorithm never leases any channel. It uses the available opportunistic channels and rejects the remaining demand. *ii) Lease when needed:* This algorithm leases channels whenever the number of active channel leases is less than the demand, provided there are channels available for leasing. Leasing a

[11] CV is the ratio of standard deviation to the mean. CV can be used as a measure of erratic nature of a trace. Higher the CV, more erratic is the trace.

[12] The problem of designing a Markov matrix whose stationary distribution has a given mean and CV can be formulated as a linear program.

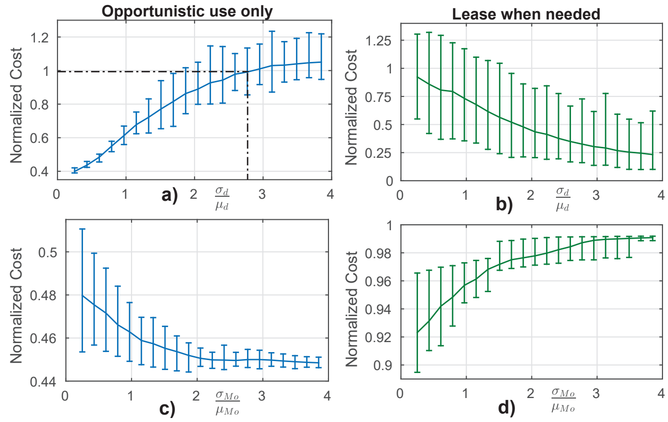

Figure 8. The performance of $\mathcal{A}_P$ with respect to two trivial algorithms. For each value of $\frac{\sigma}{\mu}$, we considered 4 values of μ and for each (σ, μ) pair, the normalized cost has been averaged over 100 traces.

channel is not advisable if the demand is erratic. This is because there is a higher probability that the demand may decrease after we lease a channel. Therefore "opportunistic use only" works better when the demand is erratic (Figure 8.a.) and "lease when needed" works better when demand is smooth (Figure 8.b.). If the number of available opportunistic channels is erratic, it is better to lease a channel because there may not be opportunistic channels available in the future. This intuition is validated by Figures 8.c. and 8.d. Figure 8 shows that $\mathcal{A}_P$ outperforms these trivial algorithms except when $\frac{\sigma_d}{\mu_d} \geq 2.95$.

Comparison with statistics based online algorithms: To implement $\mathcal{A}_P$ we do not require any knowledge of the statistics of the six traces. Therefore $\mathcal{A}_P$ will be desirable in the early stages of the deployment of 3-TSF because knowledge of market statistics will be limited or none[13]. We illustrate the advantage of $\mathcal{A}_P$ by comparing it with two statistics based algorithms: *i) Markov Decision Process (MDP):* This algorithm was proposed in [4]. It is a state-of-the-art work and its mathematical abstraction is similar to our work. MDP needs complete knowledge of the Markov matrices of all the traces. It can be implemented online only if $T \leq \tau$[14]. In our case $T > \tau$ and hence we use the following heuristic. We divide the optimization horizon T into $\frac{T}{\tau}$ frames and apply the algorithm to each frame separately. *ii) Static Leasing Strategy:* This algorithm uses the stationary distribution of the traces to compute the number of active leases required to minimize the expected cost. It then tries to maintain the optimal number of active leases subject to lease availability. Performance of such algorithms is prone to error in the statistical model. Figure 9 shows the normalized cost when μ_d is erroneous. As shown in Figure 9, $\mathcal{A}_P$ performs better than both the algorithms if there is an error of $\pm 50\%$ in μ_d. In this simulation, all statistical parameters except μ_d are known accurately. Also due to the high time complexity of MDP, we could only simulate for $\tau = 1$ week.

[13]Statistics based algorithms like [4] will outperform $\mathcal{A}_P$ if market statistics is sufficiently accurate. Accurate market statistics will be available after the 3-TSF is in operation for a sufficiently long time.

[14]The MDP based algorithm has a linear time complexity if $T \leq \tau$ and pseudo-polynomial for $T > \tau$. pseudo-polynomial time complexity is too high to be implementable online.

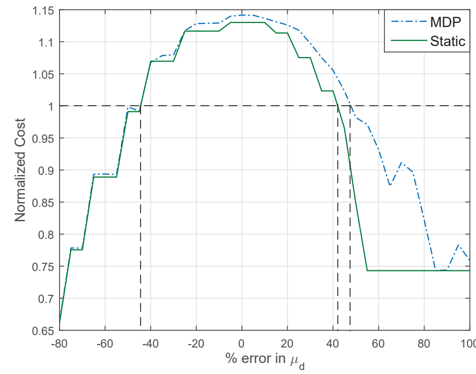

Figure 9. The performance of $\mathcal{A}_P$ with respect to two statistics based algorithms. For each % error in μ_d, the normalized cost has been averaged over 100 traces.

Remark 6 (Robustness of MDP): One may argue that $\pm 50\%$ error margin in μ_d is large and hence MDP is quite robust. However this is only due to statistical modeling error of the random process d_t. There are other random processes like M_t^o, λ_t, p_t and v_t which may also be subject to statistical modeling error. The statistical error margin for each random process may significantly decrease if we consider the cumulative effect of all the random processes. Due to high time complexity of MDP, we could not study the cumulative effect of all the random processes on its performance.

Comparison with optimal algorithm: As discussed in Remark 2, the optimal algorithm for $OP3$ (and hence $OP1$) is an offline algorithm based on dynamic programming. It has pseudo-polynomial time complexity and hence very difficult to simulate. We therefore simplify $OP3$ as follows: $M_t^o = 0\,; \forall t$, $\lambda_t = 0\,; \forall t$ and $\beta_t = \infty\,; \forall t$ (and hence $f_t(x) = 0$). With these simplifications, the renting function $F_t(r_t) = p_t r_t$. This simplifies $OP3$ to a Linear Integer Program which can be solved using standard IP solvers. We could only simulate for $\tau = 1$ week, because even the standard IP solvers have high time complexity. For this entire simulation we use a common trace of M_t^l. The moving average of M_t^l is shown in Figure 10.a. For this trace of M_t^l, $\frac{\mu_l}{M} \approx \frac{8.8}{50} = 0.18$. We conduct four simulations. In the first two simulations, the lease price P is constant at $\frac{\tau p_M}{5}$ and we vary the CV of demand d_t. In one of the simulations, the operator wins a channel with probability 0.25 and in the other it wins with probability 0.75. For $\frac{\mu_l}{M} = 0.18$ and $P = \frac{\tau p_M}{5}$, the CR is 5 as shown in Figure 7.b. As shown in Figure 10.b. and 10.c., the normalized cost[15] of $\mathcal{A}_P$ is much lower than CR. Therefore $\mathcal{A}_P$ performs much better in practice. Comparing Figure 10.b. and 10.c. we also note that the performance of $\mathcal{A}_P$ is not too sensitive to the channel winning probability of the operator.

In the third simulation we study the effect of time varying lease price P_t. P_t is assumed to be a DTMC with 50 states equally spaced in the period $\left[\frac{\tau p_M}{10}, \frac{3\tau p_M}{10}\right]$. The mean of P_t is kept fixed at $\mu_P = \frac{\tau p_M}{5}$ and the CV $\frac{\sigma_P}{\mu_P}$ is varied. Figure 7.d. shows that the normalized cost increases with CV. This observation can be explained as follows. Variation of P_t may lead to the following scenarios: P_t decreases (increases) in

[15]In a loose sense, CR is the suprema of the normalized cost.

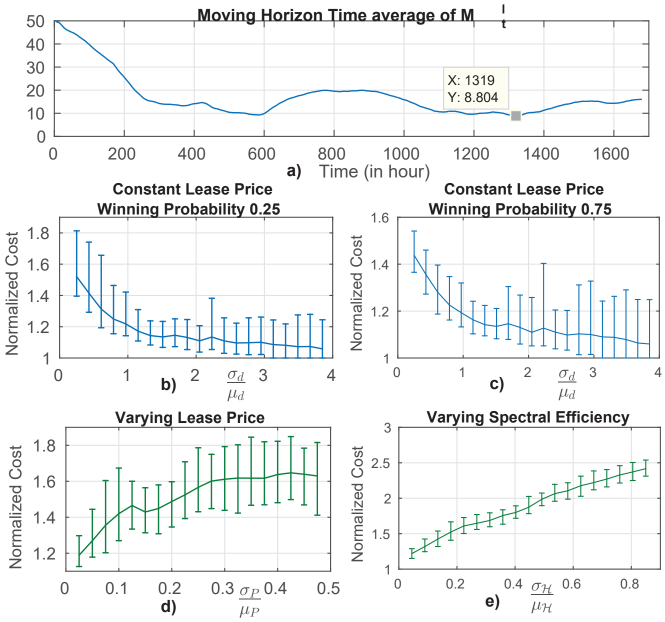

Figure 10. (a) The moving average of M_t^l, $\frac{\sum_{t=t_o-\tau+1}^{t_o} M_t^l}{\tau}$, used in this simulation. (b) \$ (c): The performance of $\mathcal{A}_P$, for constant lease price, with respect to optimal algorithm. The winning For each value of $\frac{\sigma_d}{\mu_d}$, we considered 4 values of μ_d and for each (σ_d, μ_d) pair, we averaged the normalized cost over 100 traces. (d) \$ (e): The performance of $\mathcal{A}_P$, for varying lease price (varying spectral efficiency), with respect to optimal algorithm. μ_d and σ_d is kept fixed at 4 and 1.5 respectively. For each value of $\frac{\sigma_P}{\mu_P}$ $(\frac{\sigma_{\mathcal{H}}}{\mu_{\mathcal{H}}})$, the normalized cost has been averaged over 100 traces.

future epochs. In such a case, the optimal algorithm will lease later (now). Probability of such scenarios increases as P_t becomes more erratic, i.e. the CV of P_t increases. Since $\mathcal{A}_P$ has online knowledge of P_t, it cannot make such decisions and hence underperforms as CV of P_t increases. Similar results are found when spectral efficiency $\mathcal{H}_t$ is time varying. $\mathcal{H}_t$ is assumed to be a DTMC with 10 states equally spaced in the period $[1, 10]$. Mean of $\mathcal{H}_t$ is kept fixed at $\mu_{\mathcal{H}} = 5$ and CV $\frac{\sigma_{\mathcal{H}}}{\mu_{\mathcal{H}}}$ is varied. As expected, the normalized cost increases with CV. This is shown in Figure 7.e.

In the rest of this section, we study the effect of a few trace parameters on the performance of $\mathcal{A}_P$. In this regard, we use the following definition of normalized cost: "Cost incurred by $\mathcal{A}_P$ when it uses the opportunistic channels to the cost incurred by $\mathcal{A}_P$ when it does not use the opportunistic channels." This normalized cost is a measure of the value of available opportunistic channels. Lower normalized cost implies higher value of the available opportunistic channels.

Effect of quality factor β_t: We conducted two simulations to understand the effect of quality factor β_t. In our first simulation, we study the effect of the mean quality factor μ_β on the normalized cost. As μ_β increases, the available opportunistic channels become more valuable. Hence, the normalized cost decreases with increase in μ_β as shown in Figure 11.a.

In our second simulation, we study the effect of the erroneous prediction of quality factor β_t. Implementation of $\mathcal{A}_P$ relies on computing $f_t(o)$ which in turn relies on the knowledge of β_t. The quality factor β_t depends on channel

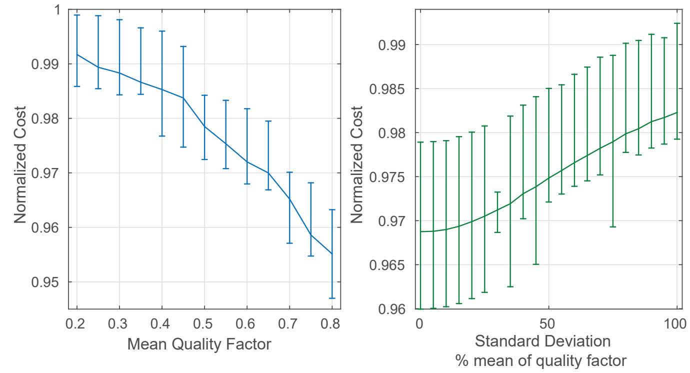

Figure 11. **a)** The effect of mean quality factor μ_β on the normalized cost. The normalized cost for each value of μ_β is averaged over 100 traces. Standard deviation of β_t, σ_β, was held constant at 0.35 throughout the simulation. **b)** Effect of erroneous β_t on the normalized cost. The normalized cost for each value of standard deviation is averaged over 100 traces.

states like number of users in a given channel, the transmission power of individual users, etc. The operator does not have direct access to this information, it can only infer it (possibly through customer feedback). Hence β_t is prone to error. Understanding the effect of erroneous β_t on the normalized cost (same as defined before) is important. To do this we add White Gaussian noise with zero mean to β_t and compute the normalized cost incurred by $\mathcal{A}_P$ as we increase the standard deviation of the Gaussian noise. This is shown in Figure 11.b. As expected, the normalized cost increases. More importantly, with standard deviation as high as 100% of the mean of the quality factor, we can still reduce the incurred cost by 1.75% if we use the available opportunistic channels.

V. Conclusion

For a wireless operator that operates in T2 and T3 of the Three-Tier Spectrum Sharing Framework, it is important to strategically decide the amount of demand to accept/reject, amount of demand to serve using opportunistically available channels (T3) and the number of channels to lease (T2), in order to minimize the total cost. Such decisions rely on demand and channel availability patterns which can be considered as random processes. In this paper, we used tools from ski-rental literature to design an algorithm that makes online decisions without any knowledge of the statistics of the involved random processes. We argue that our algorithm will be of importance in the early stages of the deployment of Three-Tier Spectrum Sharing Framework because the operator will have either limited or no knowledge of market statistics. Our algorithm has bounded competitive ratio which is nearly optimal when compared with the least possible competitive ratio. In the process of designing an online algorithm for leasing channels, we formulated and studied the modified ski-rental problem which is the state-of-the-art in ski-rental literature.

We are interested in addressing the following three issues in later works. First, the online algorithm for leasing channels which we designed has sub-optimal competitive ratio. We are interested in designing an online algorithm which is optimal in the sense of competitive ratio. Second, we are interested in designing randomized online algorithms for leasing channels. Third, we would like to explore other assumptions, like the lower bound on the moving average of the number of channels available for leasing, through which we can derive a better bound on the competitive ratio.

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