ECSE 1010

## Class #15: Matrix Characteristics

**Purpose:** The objective of this experiment is to begin to become more familiar with matrix properties and matrix mathematics.

Background: Before doing this experiment, students should be able to

- Solve 2x2 and 3x3 matrices by hand.
- Use Matlab to solve the matrix equation, $Ax = b$.
- Perform DC simulation of circuits using LTspice
- Build simple circuits consisting of a combination of resistors on protoboards.
- Make differential voltage measurements using Analog Discovery and Waveforms.
- Review the background for the previous experiments.

Learning Outcomes: Students will be able to

- Find the determinant of 2x2 and 3x3 matrices.
- Find the inverse of 2x2 and 3x3 matrices.
- Use matrix mathematics to find source voltages in constrained design problems.

Helpful links for this experiment can be found on the course website under Class #15.

## Pre-Lab

*Required Reading:* Before beginning the lab, read over and be generally acquainted with this document and the other **required reading** materials.

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## Part A – Matrix Multiplication

### Background

In our introduction to matrix mathematics, we discussed the relationship between linear system of equations and the equivalent matrix expression, $Ax = b$. We indicated that we could use matrix multiplication to recover the system of equations from the matrix expression. Matrix multiplication always results in a new matrix. If we consider two matrices, $A$ and $B$, the rules of matrix multiplication are very straightforward. In general form we can consider a new matix, $C = AB$.

1) If matrix $A$ has N rows and matrix $B$ has M columns, then matrix C will have N rows and M columns
2) The $(i,j)$ element of matrix C ($i$th row, $j$th column) is found by multiply each element of row $i$ by the corresponding element of column $j$ and summing up each of the element multiplications

As an example, consider the two matrices

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & -2 \end{bmatrix}$$

Step 1: Multiplying the matrices, $C = AB$, will result in a 2x2 matrix since $A$ has two rows and $B$ has two columns.
Step 2: Each element of matrix C is identified

Element (1,1) is determined by multiplying row 1 of $A$ by column 1 of $B$ and summing the total

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & -2 \end{bmatrix}$$

C(1,1) = (1)(2)+(2)(3)+(1)(1) = 9

Element (2,1) is determined by multiplying row 2 of $A$ by column 1 of $B$ and summing the total

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & -2 \end{bmatrix}$$

C(1,1) = (0)(2)+(-1)(3)+(1)(1) = -2

Element (1,2) is determined by multiplying row 1 of $A$ by column 2 of $B$ and summing the total

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & -2 \end{bmatrix}$$

C(1,1) = (1)(1)+(2)(1)+(1)(-2) = 1

Element (2,2) is determined by multiplying row 2 of $A$ by column 2 of $B$ and summing the total

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 1 & -2 \end{bmatrix}$$

C(1,1) = (0)(1)+(-1)(1)+(1)(-2) = -3

The new matrix $C$ is then

$$C = \begin{bmatrix} 9 & 1 \\ -2 & -3 \end{bmatrix}$$

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**Application of matrix multiplication**

Determine the dimensions (size) of each matrix and the elements in that matrix

a) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, $C = AB$

b) $D = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $F = DE$

c) $D = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $G = ED$

For the design problem in experiments 12 and 13, we could have used matrix multiplication to help determine the source voltage levels. For example,

d) Based on the resistors in your Design Problem 3 circuit from experiment 13, apply KCL as if the nodal voltages were unknown and determine the matrix expression $Ax = b$. Leave the voltage sources in symbolic form as V1 and V2. (In other words, use your known resistor values to build the *A* matrix, but the *b* terms should include V1 and V2 as unknown values.). Given that the values of vector *x* are known in the design problem (they are nodal voltage values specified in the problem), we can treat *A* and *x* as matrices and apply matrix multiplication to get a new matrix (the *b* vector). Complete the matrix multiplication to determine the numerical values of the *b* vector. Use that result to determine the source voltage values that would be needed to meet the design specifications. Do your results agree with your design from experiment 13?

*Using Matab*

To complete matix multiplication in Matlab, it is as simple as multiplication between two decimal numbers. For example, for the first problem, you can simply enter

```
>> C = A*B
```

and Matlab will provide the result

e) Use Matlab to verify your answers to problems a-c. Include image captures of your Matlab results.

## Part B – The Determinant of a Matrix

### Background

We have used matrix mathematics to find the voltages at nodes in a circuit. The form of the expression is the familiar, $Ax = b$, where $A$ is a square matrix, $x$ is the column vector of unknown values, and $b$ is the column vector of source terms. This expression is a fundamental part of engineering modelling, especially when using computer simulations. However, the expression only has a solution if the system of equations is linearly independent.

For example, consider the following two equations with two unknowns,

$$2x_1 + 1x_2 = 1 \qquad (1)$$

$$-1x_1 - 1x_2 = 1 \qquad (2)$$

We can solve the linear system, finding that $x_1 = 2$ and $x_2 = -3$.

Now consider a second set of equations,

$$2x_1 + 1x_2 = 1 \qquad (3)$$

$$-4x_1 - 2x_2 = 1 \qquad (4)$$

There is not a solution to this set of equations. We see that two equations are not linearly independent, since the 'left' side of equation 4 can be obtained by multiplying the left side of equation 3 by -2. Equation manipulation would result in 0 = 3. In matrix form, we can write the mathematical steps as

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Multiplying row 1 by 2 and adding to row 2, we get

$$\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

We see that this matrix has a row of zeros. A matrix with a row of zeros or column of zeros has a **determinant** of zero. The determinant of a matrix is a very important property indicating whether there is a unique solution for each unknown. **If the determinant does not equal zero, a unique solution to the linear system exists. On the other hand, if the determinant is zero, then an infinite number of solutions exist.**

Rather than perform matrix manipulation to find out whether a column or row of zeros occurs, there are techniques to find the determinant of a matrix. These techniques become very important when solving problems with large matrices. We will limit our studies to 2x2 matrices and use established forumulas (ie. you can easily find these techniques online).

### Application to 2x2 matrices

For a 2x2 matrix, we will use the first set of above linear equations as an example.

$$\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The determinant of matrix $A$ is found using the two diagonals, multiplying the two terms on the diagonal, and then subtracting the 'forward' diagonal from the 'backward' diagonal.

$$\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = (2)(-1) - (1)(-1) = -1$$

We can generalize the expression to

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a)(d) - (b)(c)$$

Looking at the second set of equations and the corresponding matrix

$$\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\det\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} = (2)(-2) - (1)(-4) = 0$$

giving the zero determinant we discussed previously.

**Problems**

Determine the determinant of the following 2x2 matrices.

f) $\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

g) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

h) $\begin{bmatrix} 2 & 0 \\ -1 & 0 \end{bmatrix}$

For the circuit problems we have analyzed, there have always been solutions when finding the nodal voltages after applying KCL at each unknown node.

i) Revisit Design Problem 2 in experiment 12 and determine the matrix expression associated with nodal analysis. Find the determinant of the expression and verify that there is a solution (that the determinant is non-zero)

**Application to 3x3 matrices**

For a 3x3 matrix, the process is similar, using 2x2 submatrices to break the problem down. A submatrix is a smaller matrix within a larger matrix. There are several equivalent approaches. We will use the elements of row 1 to build submatrices. Considering matrix *A*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & -2 \end{bmatrix}$$

We obtain a submatrix for a particular element by 'crossing out' the row and column associated with that element and keeping 'what is left'. For example, the submatrix associated with A(1,1) is

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & -2 \end{bmatrix}, \text{ giving submatrix } \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

Likewise, for A(1,2) and A(1,3), we get submatrices

$$A=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 1\\ 2 & 1 & -2\end{bmatrix}\text{, giving submatrix }\begin{bmatrix}0 & 1\\ 2 & -2\end{bmatrix}$$

$$A=\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 1\\ 2 & 1 & -2\end{bmatrix}\text{, giving submatrix }\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}$$

Having found the submatrices associated with each element of row 1, we can now find the determinant by multiplying each element by the determinant of its submatrix. Additionally, we need to multiply by $(-1)^{i+j}$ by each term, where *i* and *j* are the row number and column number of each element. For matrix A about, the determinant is found to be

$$\det(A)=\det\begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 1\\ 2 & 1 & -2\end{bmatrix}=(-1)^{1+1}(1)\det\begin{vmatrix}1 & 1\\ 1 & -2\end{vmatrix}+(-1)^{1+2}(2)\det\begin{vmatrix}0 & 1\\ 2 & -2\end{vmatrix}+(-1)^{1+3}(-1)\det\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix}$$

$$\det(A)=(1)(1)(-3)+(-1)(2)(-2)+(1)(-1)(-2)$$

$$\det(A)=3$$

**Problems**

Determine the determinant of the following 3x3 matrices.

j) $A=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & -1\\ 1 & 1 & 1\end{bmatrix}$

k) $B=\begin{bmatrix}1 & 1 & 2\\ 1 & 0 & -1\\ 2 & 1 & 1\end{bmatrix}$

For the circuit problems we have analyzed, there have always been solutions when finding the nodal voltages after applying KCL at each unknown node.

l) Revisit Design Problem 3 in experiment 12 and determine the matrix expression associated with nodal analysis. Find the determinant of the expression and verify that there is a solution (that the determinant is non-zero)

*Matlab*

The determinant can be found in Matlab use the 'det' command

```
>> det(A)
```

where A is a matrix.

m) Use Matlab to verify your answers to the above problems j and k. Include an image capture of your results.

## Part C – The Inverse of a Matrix

**Background**
The Identity matrix is defined as a matrix that has 1s on the main diagonal and 0s for all other elements. For example, the 4x4 identify matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In algebra, we know that if we have some variable *x* and we divide that variable by itself, we get 1. We can write that expression mathematically as

$$x^{-1} * x = 1$$

In matrix mathematics, a similar relationship holds for the Identify matrix. The inverse of a matrix multiplied by that matrix results in the Identify matrix. This expression is written as

$$A^{-1} * A = I$$

Finding the inverse of the matrix is again a very important part of analyzing linear systems. There are several approaches. We will consider a method that is similar to that discussed when solving the expression $Ax = b$. In this case, we will start with both our original matrix *A* and the Identity matrix, *I*. Our goal is to use row addition and multiplication to reduce the matrix *A* to the Identify matrix, while at the same time performing the exact same operation on the identify matrix.

Using the same 3x3 matrix from the last section,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 2 & 1 & -2 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We will write the two matrices side by side

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right]$$

Multiplying row 1 by -2 and adding it to row 3, we 'remove' the 2 at element position (3,1). We need to do the exact same operation to the identity matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -3 & 0 & -2 & 0 & 1 \end{array}\right]$$

Multiplying row 2 by 3 and adding it to row 3, we 'remove' the -3 at element position (3,2). Again, we need to do the exact same operation to the identity matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 3 & -2 & 3 & 1 \end{array}\right]$$

Dividing row 3 by 3, again doing the same operation on the identity matrix

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & \frac{-2}{3} & 1 & \frac{1}{3} \end{array}\right]$$

Multiplying row 3 by -1 and adding it to row 2. Also, adding row 3 to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{1}{3} & 1 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{-2}{3} & 1 & \frac{1}{3} \end{array}\right]$$

Finally, multiplying row 2 by -2 and adding to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{-2}{3} & 1 & \frac{1}{3} \end{array}\right]$$

The inverse of A is now found to be

**Problems**

Determine the inverse of the following 3x3 matrices.

n) $A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$

o) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

p) $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$ (oops, why?)

*Matlab*

The inverse can be found in Matlab use the 'inv' command

```
>> inv(A)
```

where A is a matrix.

q) Use Matlab to verify your answers to the above problems n and o. Include an image capture of your results.

r) Revisit Design Problem 3 in Experiment 12 and determine the inverse of the matrix expression associated with nodal analysis (you already used the matrix in Part B). You can use Matlab to find the inverse. For your chosen source voltages, mathematically verify that the specified nodal voltages were met by completing the matrix mathematics $A^{-1} * b = x$