



**Class #10: Experiment 10**  
**Nodal Analysis and Matrix Solutions**

In this experiment we will develop the methods to set up the simultaneous expressions needed to determine nodal voltages. We will then implement and solve the equations using matrices.

*Be sure that one of your team reads the entire write-up before beginning your experiments.*

**Purpose:** The objective of this experiment is to learn more about energy storage, steady state analysis and basic filters.

Background: Before doing this experiment, students should be able to

- Use Ohm's Law to determine the voltage across, current through and/or resistance of a resistor when given information on the other two parameters.
- Determine the total resistance of a small number of resistors in series or parallel
- Determine the voltage across a resistor using voltage divider concepts

Learning Outcomes: Students will be able to

- Apply KCL to obtain a linearly independent equation.
- Build the system of equations needed to determine the voltage at nodes in a circuit.
- Solve 2x2 and 3x3 matrix mathematics 'by hand'
- Run Matlab and use it to solve basic mathematics

**Pre-Lab**

*Required Reading:* Before beginning the lab, read over and be generally acquainted with this document and the other **reading and viewing** materials listed under Class #7 on the ECSE 1010 website.

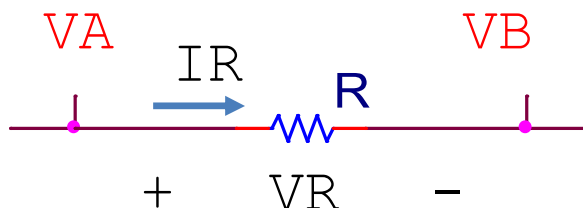
**Software**

Matlab needs to be installed on your laptop. The university has an academic license for Matlab available to students. The installation requires a request for a license and can be a time consuming process. Documentation to install Matlab may be found at the following link,

<https://dotcio.rpi.edu/services/software-labs/software-documentation/matlab-campus-wide-install>

## Background

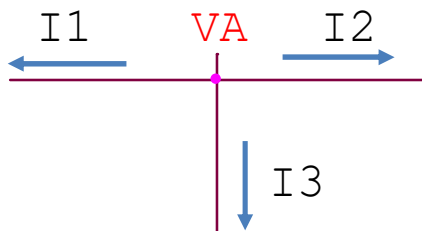
*Voltage Differences and Ohm's Law:* In previous experiments we have discussed the relationship between voltage and current in a resistor (Ohm's Law). In the last experiment we used nodal analysis to determine the voltage across a resistor (or combination of resistors). The voltage we use in Ohm's Law is always a voltage difference, ie. the voltage on one side of the resistor minus the voltage on the other side of the resistor.



**Figure A-1: Resistor Voltage and Current**

In Figure A-1, the voltage across the resistor is  $V_R = V_A - V_B$ , as we saw in the previous laboratory. Likewise, the current through the resistor is  $I_R = V_R/R$ . We can then use the voltage difference expression and rewrite the current through the resistor as,  $I_R = \frac{V_A - V_B}{R}$ . This form of the equation is used to set up a system of equation to find the voltage at every node in a circuit.

Important note: Current in a resistor has direction and will always flow from a higher voltage node to a lower voltage node. In Figure A-1, the '+' side of the resistor is assumed to be a higher voltage, indicating that current will flow to the 'right'. These designations are called the **polarity** of the voltage and current. It is possible that VB is a higher voltage than VA, which would result in VR being negative (using the expression  $V_R = V_A - V_B$ ). If the voltage across a resistor is negative, then the current will also be negative since we always use the same polarity designations.

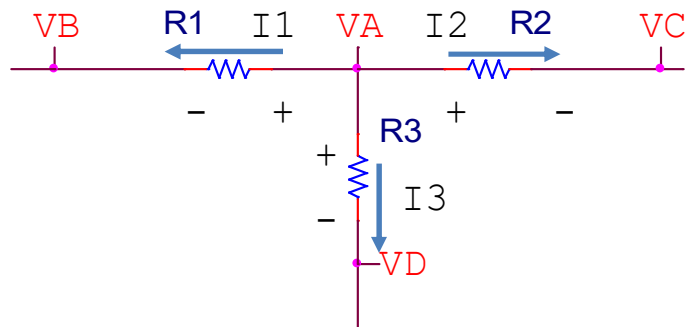


**Figure A-2: Currents Leaving Node VA**

*Kirchoff's Current Law (KCL):* In words, KCL indicates that the total amount of current entering a node must be equal to the total amount of current leaving a node. In other words, there is no accumulated charge (electrons) at a node. Visually, we can represent that concept in Figure A-2. Node VA has three paths (connections to various components). If we draw currents leaving the node, as shown in the figure, we can express KCL mathematicall as,

$$I_1 + I_2 + I_3 = 0$$

In this expression, we notice that at least one of the currents must be negative. In other words, at least one of the currents must be entering the node. As mentioned above, a negative current just means we 'guessed' the wrong direction when we drew the arrow.



**Figure A-3: KCL Applied at a Node**

We can apply KCL to a node connected to resistors, as shown in Figure A-3. Again, we assume currents are leaving the node. Based on that current polarity, we must assign a consistent voltage polarity, with the + side on the VA side of each resistor and the – side on the other node for each resistor (VB for R1, etc.). Applying KCL, we get the same expression as seen on the previous page,

$$I_1 + I_2 + I_3 = 0$$

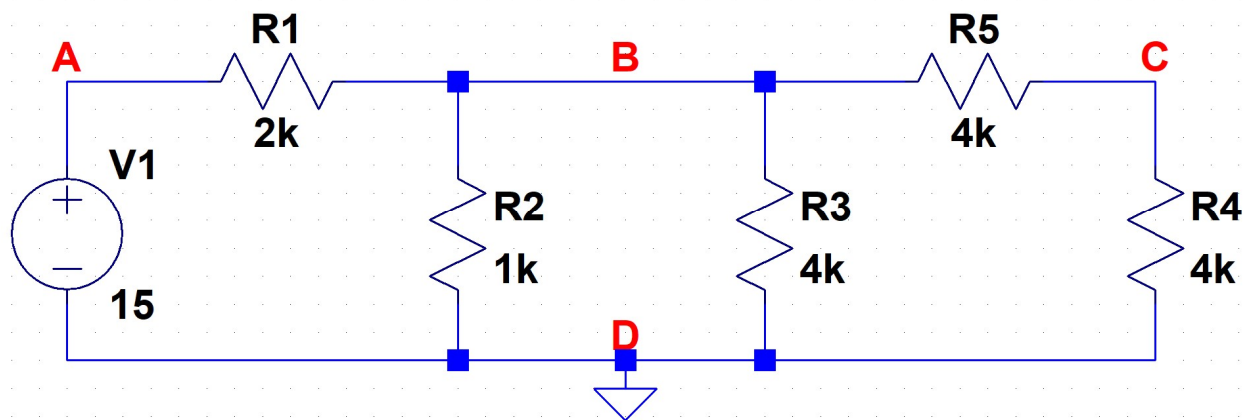
We can now apply Ohm’s Law to each of those currents and use voltage difference concepts to include the nodal voltages,

$$I_1 = \frac{V_A - V_B}{R_1} \qquad I_2 = \frac{V_A - V_C}{R_2} \qquad I_3 = \frac{V_A - V_D}{R_3}$$

We can substitute these Ohm’s Law expressions back into the KCL expression, to obtain an equation

$$\frac{V_A - V_B}{R_1} + \frac{V_A - V_C}{R_2} + \frac{V_A - V_D}{R_3} = 0$$

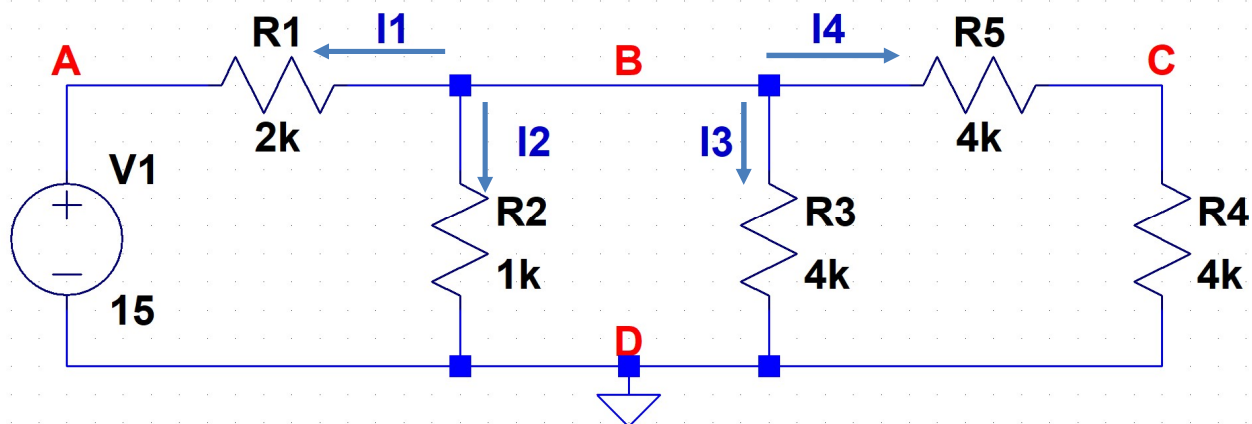
We now have one equation with four unknowns (VA, VB, VC, and VD). In order to find the values of each of those nodal voltage, we would need three more **linearly independent** equations. We will use an example from the previous lab.



**Figure A-4: More Complicated Example of Nodal Connections**

In the previous lab, the circuit in Figure A-4 was used as an example of assigning nodes. There four nodes with seemingly unknown voltages. Normally, this would mean we need four equations to find those nodal voltages. However, an important aspect from the previous laboratory is that we always need a ground reference. In the Figure A-4, we have taken the next step and chosen node D as ground. Once we assign a ground, the voltage at that node is then designated as 0V and it is no longer an unknown voltage. Now that we have a ground, we can also take advantage of the fact that a voltage source is a voltage difference and we then know that  $V_1 = V_A - V_D$ . Substituting in numbers,  $15 = V_A - 0$ , giving  $V_A = 15V$ . The voltage at A is now known as well, leaving us with two unknown voltages at nodes

B and C. To find the voltages at those nodes, we need to use KCL at each of those nodes to find a linearly independent set of equations (in this case just two equations for two unknowns). Following the same process at the start of this section, we can apply KCL at node B, assuming all currents are leaving the node.



**Figure A-5: Currents Leaving Node B**

KCL at node B gives

$$I_1 + I_2 + I_3 + I_4 = 0$$

Applying Ohm's Law and voltage differences to each current term

$$\frac{V_B - V_A}{R_1} + \frac{V_B - V_D}{R_2} + \frac{V_B - V_D}{R_3} + \frac{V_B - V_C}{R_5} = 0$$

Recalling that we know the values for  $V_A$  and  $V_D$ , we can then rewrite the above equation as

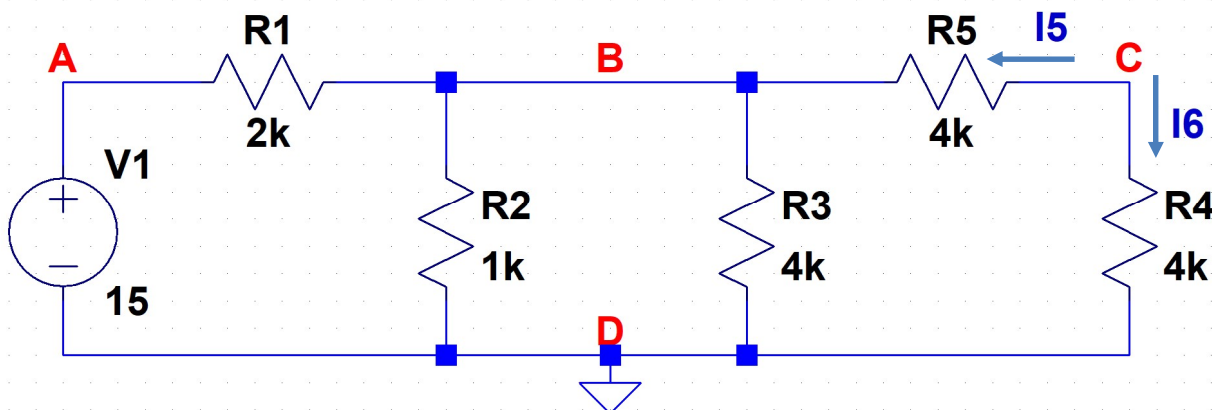
$$\frac{V_B - 15}{2000} + \frac{V_B - 0}{1000} + \frac{V_B - 0}{4000} + \frac{V_B - V_C}{4000} = 0$$

Finally, we can collect terms and rewrite the expression as

$$\left( \frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000} \right) V_B + \left( \frac{-1}{4000} \right) V_C = \frac{15}{2000}$$

Giving us one equation with two unknowns.

Repeating the process at node C



**Figure A-6: Currents Leaving Node C**

KCL at node B gives

$$I_5 + I_6 = 0$$



Applying Ohm's Law and voltage differences to each current term

$$\frac{V_C - V_B}{R5} + \frac{V_C - V_D}{R4} = 0$$

Recalling that we know the values for VA and VD, we can then rewrite the above equation as

$$\frac{V_C - V_B}{4000} + \frac{V_C - 0}{400} = 0$$

Finally, we can collect terms and rewrite the expression as

$$\left(\frac{-1}{4000}\right)V_B + \left(\frac{1}{4000} + \frac{1}{400}\right)V_C = 0$$

Giving us our second equation.

With two equations and two unknowns, we can now solve for the voltage at VB and VC

$$\left(\frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000}\right)V_B + \left(\frac{-1}{4000}\right)V_C = \frac{15}{2000}$$

$$\left(\frac{-1}{4000}\right)V_B + \left(\frac{1}{4000} + \frac{1}{400}\right)V_C = 0$$

Using methods we learned in algebra, we can find the two voltages. For example, we can determine an expression for VB using the second equation

$$V_B = 4000\left(\frac{1}{4000} + \frac{1}{400}\right)V_C$$

Substitute that into the first equation

$$\left(\frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000}\right)(4000)\left(\frac{1}{4000} + \frac{1}{400}\right)V_C + \left(\frac{-1}{4000}\right)V_C = \frac{15}{2000}$$

And solve for VC, getting VC = 2V. Substituting the value for VC back into one of the two equations and get VB = 4V. We can then find the voltage difference across any resistor and then the current through that resistor using Ohm's Law.

*Matrices:* A system of linear equations, as seen above can be solved using matrix mathematics. Initially, we need to represent the system as a combination of a matrix and vectors. The expression is written as  $Ax = b$ , where  $A$  is a matrix and  $x$  and  $b$  are vectors. A vector is a one dimension array of numbers, for example

$[1 \ 4 \ 0.5 \ -2]$  is a row vector

$\begin{bmatrix} -2 \\ 4 \\ 0.33 \end{bmatrix}$  is a column vector

A matrix is a two dimension set of numbers, for example

$\begin{bmatrix} 1 & 0 & -5 \\ 0.2 & 1.6 & 3 \\ -2 & -0.4 & 5 \end{bmatrix}$  is a square matrix, same number of rows and columns

$\begin{bmatrix} 1 & 3 & 6 \\ 0 & -1 & 1 \end{bmatrix}$  is a rectangular matrix, different number of rows and columns.

For our circuit solutions, we will be using square matrices.

To transform the system of equations into matrix form, we treat the coefficients of our unknowns as elements in the matrix  $A$ . The unknowns themselves are the elements of the array  $x$ . The constants terms (on the right side) are elements of the vector  $b$ . Repeating the two equations from above,

$$\left(\frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000}\right)V_B + \left(\frac{-1}{4000}\right)V_C = \frac{15}{2000}$$

$$\left(\frac{-1}{4000}\right)V_B + \left(\frac{1}{4000} + \frac{1}{4000}\right)V_C = 0$$

In matrix form, the same system can be written as

$$\begin{bmatrix} \left(\frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000}\right) & \left(\frac{-1}{4000}\right) \\ \left(\frac{-1}{4000}\right) & \left(\frac{1}{4000} + \frac{1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{15}{2000} \\ 0 \end{bmatrix}$$

The first row in the matrix corresponds to the coefficients of  $V_A$  and  $V_B$  in the first equation in our system of equations. Likewise, the second row in the matrix corresponds to the coefficients of  $V_A$  and  $V_B$  in the second equation in our system of equations. To recover our system of equations, we can use matrix multiplication, which multiplies the elements of each row times the elements of the unknown column and sets that equal to the element in the corresponding row of the  $b$  vector. For example, using the above matrix,

$$\begin{bmatrix} \left(\frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000}\right) & \left(\frac{-1}{4000}\right) \\ \left(\frac{-1}{4000}\right) & \left(\frac{1}{4000} + \frac{1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{15}{2000} \\ 0 \end{bmatrix}$$

Multiplying the first element of the first row by  $V_A$  and the second element of the first row by  $V_B$ , adding them and setting the expression equal to the first element of the right side column vector gives the expression.

$$\left(\frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000}\right)V_B + \left(\frac{-1}{4000}\right)V_C = \frac{15}{2000}$$

$$\begin{bmatrix} \left(\frac{1}{2000} + \frac{1}{1000} + \frac{1}{4000} + \frac{1}{4000}\right) & \left(\frac{-1}{4000}\right) \\ \left(\frac{-1}{4000}\right) & \left(\frac{1}{4000} + \frac{1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{15}{2000} \\ 0 \end{bmatrix}$$

Multiplying the first element of the second row by  $V_A$  and the second element of the second row by  $V_B$ , adding them and setting the expression equal to the second element of the right side column vector gives the expression.

$$\left(\frac{-1}{4000}\right)V_B + \left(\frac{1}{4000} + \frac{1}{4000}\right)V_C = 0$$

*Solving Matrix Mathematics by Hand:* The methods used to solve a system of equations are the same used to solve for the unknowns in a matrix. We can clean up the expressions by finding a common denominator.

$$\begin{bmatrix} \left(\frac{8}{4000}\right) & \left(\frac{-1}{4000}\right) \\ \left(\frac{-1}{4000}\right) & \left(\frac{2}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{30}{4000} \\ 0 \end{bmatrix}$$

There are several ways we can manipulate the matrix formulation, without changing the answer. For example, we can change the order of the rows in matrix  $A$  and vector  $b$ . The change has to be the same.

$$\begin{bmatrix} \left(\frac{-1}{4000}\right) & \left(\frac{2}{4000}\right) \\ \left(\frac{8}{4000}\right) & \left(\frac{-1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{30}{4000} \end{bmatrix}$$

Note, the vector of unknowns did not change.

We can multiply a row by a constant. For example, we can multiply the top row of the above matrix by 8, remembering multiply both the row in the  $A$  matrix and the corresponding element in the  $b$  vector.

$$\begin{bmatrix} \left(\frac{-1(8)}{4000}\right) & \left(\frac{2(8)}{4000}\right) \\ \left(\frac{8}{4000}\right) & \left(\frac{-1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 0(8) \\ \frac{30}{4000} \end{bmatrix} \rightarrow \begin{bmatrix} \left(\frac{-8}{4000}\right) & \left(\frac{16}{4000}\right) \\ \left(\frac{8}{4000}\right) & \left(\frac{-1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{30}{4000} \end{bmatrix}$$

We can combine rows by adding or subtracting them. For example, we can add row 2 to row 1 in the above matrix. Row addition, is element by element,

$$\begin{bmatrix} \left(\frac{-8}{4000} + \frac{8}{4000}\right) & \left(\frac{16}{4000} + \frac{-1}{4000}\right) \\ \left(\frac{8}{4000}\right) & \left(\frac{-1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 0 + \frac{30}{4000} \\ \frac{30}{4000} \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 0 & \left(\frac{15}{4000}\right) \\ \left(\frac{8}{4000}\right) & \left(\frac{-1}{4000}\right) \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} \frac{30}{4000} \\ \frac{30}{4000} \end{bmatrix}$$

We can now look at the first row, using the technique described previously and obtain an expression for VC.

$$\left(\frac{15}{4000}\right)V_C = \frac{30}{4000}, \text{ giving } VC = 2V.$$

Using this value of VC, we can use the second row to get an expression

$$\left(\frac{8}{4000}\right)V_B + \left(\frac{-1}{4000}\right)(2) = \frac{30}{4000} \rightarrow \left(\frac{8}{4000}\right)V_B = \frac{32}{4000}, \text{ giving } VB = 4V.$$

We can extend this process to larger matrices. A 3x3 example follows

$$\begin{bmatrix} 1 & 0 & 4 \\ 4 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$$

Subtracting the first row from the third row

$$\begin{bmatrix} 1 & 0 & 4 \\ 4 & 2 & -1 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 9 \\ 9 \end{bmatrix}$$

Multiplying the first row by -4 and adding it to the second row

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -17 \\ 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \\ 9 \end{bmatrix}$$

Multiply the second row by 1.5 (3/2) and adding it to the third row

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & -17 \\ 0 & 0 & 22.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 21 \\ -22.5 \end{bmatrix}$$

Working from the third row to the first row

$$22.5x_3 = -22.5 \rightarrow x_3 = -1$$

$$2x_2 - 17(-1) = 21 \rightarrow x_2 = 2$$

$$1x_1 - 0(2) + 4(-1) = -3 \rightarrow x_1 = 1$$



*Using Matlab:* For the matrix mathematics discussed on the previous page, Matlab is an outstanding tool that can be used to solve for the unknown variables. When considering the matrix formulation,  $Ax = b$ , we need to build the matrix  $A$  and vector  $b$  in Matlab.

Using the 3x3 example on the previous page, we build the matrix by entering each row with the individual elements separated by a spade. At the end of each row, a semicolon is used to indicated the end of a row. The command line entry, followed by the matrix is shown below (amat is the matrix A)

```
>> amat = [1 0 4; 4 2 -1; 1 3 1]

amat =

     1     0     4
     4     2    -1
     1     3     1
```

**Figure A-7: Entering a Matrix into Matlab**

Similarly, to build the column vector  $b$ , we enter each element followed by a semicolon.

```
>> b = [-3; 9; 6]

b =

    -3
     9
     6
```

**Figure A-8: Entering a Column Vector into Matlab**

We can now find the results for the three unknowns in the vector  $x$ . In matrix mathematics, the division operation is replaced by the inverse operation. For matrices, we would write  $x = amat^{-1}(b)$  where the  $amat^{-1}$  is the inverse of the matrix  $amat$ . We will discuss how to find inverses later. The good news is that Matlab will take of the mathematics. To find the unknowns, we use the following command line expression

```
>> x = inv(amat)*b

x =

    1.0000
    2.0000
   -1.0000
```

**Figure A-8: Entering a Column Vector into Matlab**

The three unknowns are presented in column vector form.



*Summarizing Nodal Analysis:*

To solve for the voltages at the nodes of a circuit, a summary of the process follows.

- 1) Label all the nodes in the circuit.
- 2) Pick a node to be common ground and set the voltage at the node to be zero.
- 3) For each voltage source, use voltage differences to find the voltage at another node (note, the voltage source needs to be connected to ground).
- 4) At each of the remaining unknown nodal voltage, apply KCL to get an equation
- 5) Implement the system of equations as a matrix expression,  $Ax = b$ .
- 6) Solve the matrix expression

## Experiment

### Part B

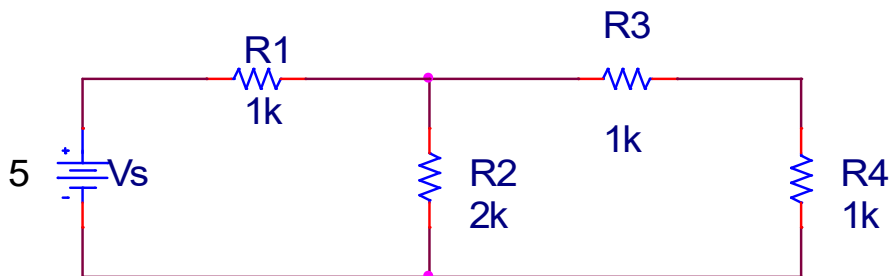


Figure B-1 Four Node Circuit

Following the process developed on previous pages (and the summary on the previous page).

- 1) In your report, include the circuit schematic with each node labelled.
- 2) Identify a ground node (it is best to pick a node connected to the source)
- 3) Use the source to identify the voltage at one of the labelled nodes. Which node and what voltage?
- 4) How many unknown voltages are left?
- 5) Apply KCL to each of those unknowns. Include the expressions in your report.
- 6) Combine your simultaneous equations into matrix form, filling in values for the matrix  $A$  and vector  $b$ .
- 7) Solve the matrix 'by hand'.
- 8) Verify your solution using Matlab for the matrix mathematics (take screen shots of your process and include them in your report).
- 9) Simulate the circuit in LTspice and check your answers. (Comparing calculations and simulations is a frequent part of engineering.)

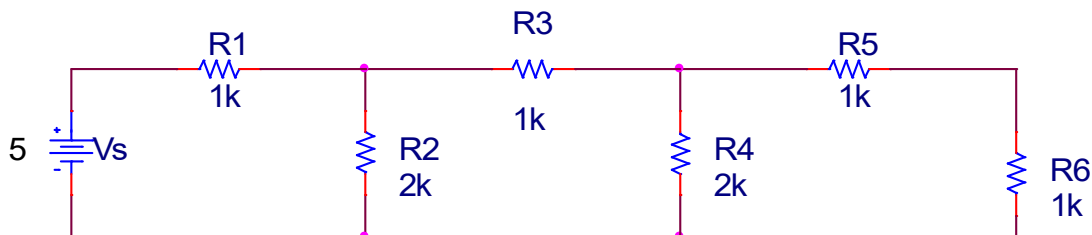


Figure B-2 Five Node Circuit

Repeat steps 1-9 for the circuit in Figure B-2. Note, the resistance of resistor R4 has changed to 2V.

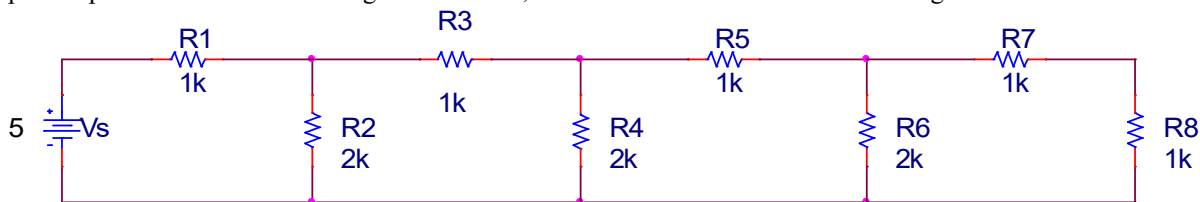


Figure B-3 Six Node Circuit

Repeat steps 1-6, 8-9 for the circuit in Figure B-3 (you don't need to solve the matrix by hand). Note, the resistance of resistor R6 has changed to 2V.

The above circuits are a ladder networks. As we change the 'terminal' resistor and add a new 'section', what do you notice about the voltage at each new node?

*Part C*

It would be possible to use voltage divider, series resistance, and parallel resistance concepts to analyze the circuits on the previous page. That is not always possible. For example, none of the resistors in Figure C-1 are in parallel or in series. No simplification is possible and we must use nodal analysis to determine the voltages. This circuit has particular properties and is called a Wheatstone Bridge (there is a Wikipedia page for this circuit).

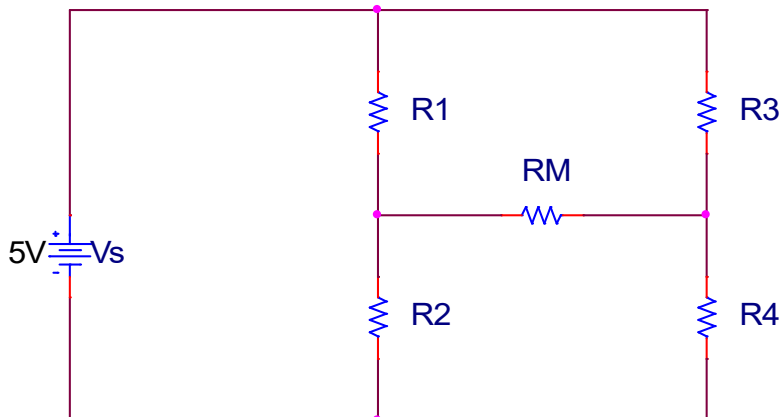


Figure C-1 Wheatstone Bridge

For the resistors values,

- R1 = 470Ω
- R2 = 1kΩ
- R3 = 1kΩ
- R4 = 470Ω
- RM = 100Ω

- Use the methods developed previously to calculate the voltage across RM.
- Build the circuit and measure the voltage across RM using the Voltmeter
- Do the results agree?
- Replace RM with 100kΩ. Did the measured voltage change?

The above resistor values make the bridge unbalanced. A balanced Wheatstone Bridge's resistors have the relationship  $\frac{R_1}{R_2} = \frac{R_3}{R_4}$ . If the bridge is balanced, the voltage across RM will be zero and independent of the value of RM.

For the resistors values,

- R1 = 470Ω
- R2 = 1kΩ
- R3 = 4.7kΩ
- R4 = 10kΩ
- RM = 100Ω

- Using nodal analysis, verify that the bridge is balance (that the two nodes on each side of RM have the same resistance).
- Build the circuit and check your result experimentally. Note, there is always error due to equipment, resistor tolerances, etc., so it would be very surprising if the measured result was exactly zero.
- Does replacing RM with a 100kΩ resistor affect the measurement?