AC Steady State Analysis

What if we are given a circuit and we wish to find the output of that circuit for a range of inputs? We could wire the circuit, attach the function generator to the input, and hook up the scope to the output. Then, we could alter the input systematically and observe how the output responds to these changes. If we could use this information to find a mathematical function that relates the input to the output, then we could use the function to predict how the circuit will behave for any given input, as shown in figure 1 below. What if we could use the components in the circuit to derive this function? Then we could predict the output of any circuit for any given input.
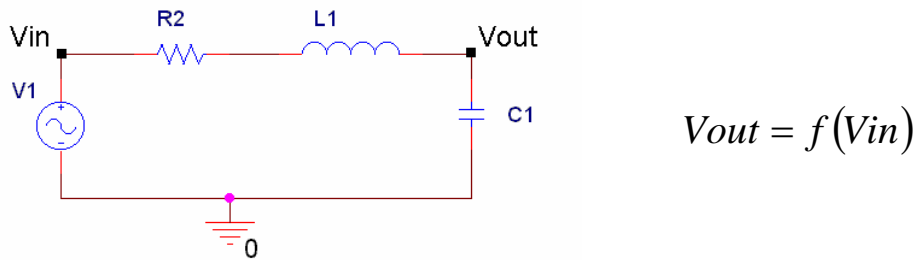

$$Vout = f(Vin)$$

Figure 1

A. What is a Transfer Function?

A transfer function relates the output and the input of a circuit. In order for a transfer function to be useful, it must be simple to use and easy to find using the circuit diagram. Therefore, let us define a transfer function, H, as the ratio of the output to the input of a circuit.

$$H \equiv \frac{V_{out}}{V_{in}} \quad \text{[equation 1]}$$

In a circuit containing only resistors, transfer functions can be easily found using voltage dividers. In figure 2, a voltage divider is used to find the transfer function at Vout.
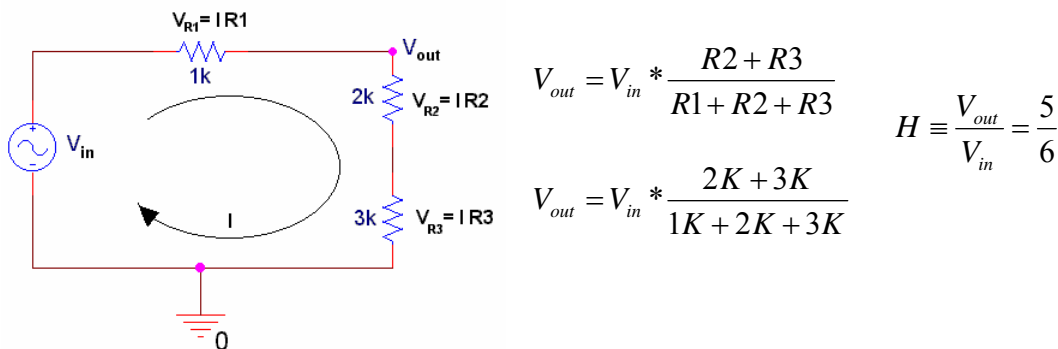

$$V_{out} = V_{in} * \frac{R2+R3}{R1+R2+R3}$$

$$V_{out} = V_{in} * \frac{2K+3K}{1K+2K+3K}$$

$$H \equiv \frac{V_{out}}{V_{in}} = \frac{5}{6}$$

Figure 2

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Since H is 5/6 for this circuit, we can find the output at Vout by multiplying any input, Vin, by 5/6. What if the circuit contains components other than resistors?
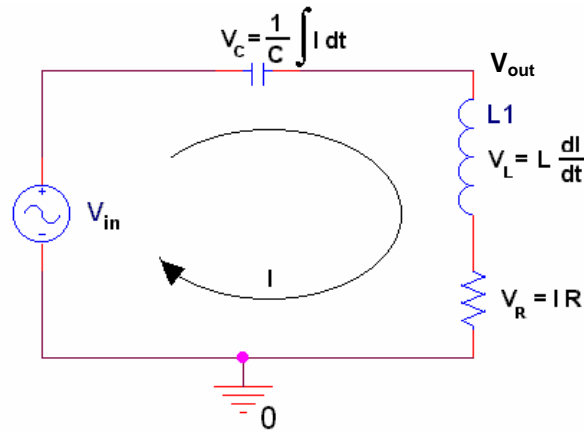

Figure 3

The equations that govern the behavior of capacitors and inductors are not linear, like those for resistors. Voltage dividers are based on these simple linear relationships. How can we find a simple transfer function for a circuit with voltage relationships that have derivatives and integrals? We could use differential equations, but that would not be the simple solution we are looking for. We need to find a way to get rid of these integrals and derivatives so that we can get back to a simple linear relationship. This is the fundamental motivation for steady state analysis.

B. A Model for Steady State Analysis

Steady state analysis is a way to analyze AC circuits mathematically. We can do this using linear transfer functions (and without the use of differential equations) if we do two important things. The first is to take advantage of the fact that AC signals are sinusoids that do not change in frequency. The second is to map the input from the time domain into a domain based on the amplitude and frequency of the signals involved.

B.1 Steady State Sinusoids

The term *steady state* refers to the concept that a circuit, when attached to an AC input signal, will, after a finite period of time, reach a state in which the signal across each device in the circuit behaves like a sinusoid of the same frequency as the input. The signal across each device will have a constant amplitude and phase relative to the input that does not change once steady state has been reached. Once a circuit has reached steady state, the output at any point in the circuit will look like a sinusoid with the same frequency as the input, a constant amplitude, and a phase that does not change relative to the input phase. It is this feature that we will exploit to find transfer functions.
Sinusoids have a special mathematical property in that the derivative of a sinusoid is just another sinusoid with the same frequency, but shifted in phase. The same is true for the

integral os a sinusoid. So when we need to take the derivative (or integral) of a sinusoidal signal, we can think of it simply as the application of some change in amplitude and some shift in phase.

Mathematically, the derivative of a sine wave of frequency ω and amplitude A, $Vin = A\sin(\omega t + \phi)$, is given by a cosine wave of frequency ω and amplitude $A\omega$, $\dfrac{dVin}{dt} = A\omega\cos(\omega t + \phi)$. The output of the derivative operation is a new sine wave with a different amplitude and a phase shifted by and additional 90 degrees, $\dfrac{dVin}{dt} = A\omega\sin(\omega t + \phi + \pi/2)$. Mathematically, this means that $\dfrac{dVin}{dt} = A'\sin(\omega t + \phi')$ where $A' = A\omega$ and $\phi' = \phi + \pi/2$. The derivative of a cosine (or any other sinusoid) results in the same changes. For example, if $Vin = A\cos(\omega t + \phi)$, then

$$\frac{dVin}{dt} = -A\omega\sin(\omega t + \phi) = A\omega\sin(\omega t + \phi + \pi) = A\omega\cos(\omega t + \phi + \pi - \pi/2) = A\omega\cos(\omega t + \phi + \pi/2)$$

and, therefore, $\dfrac{dVin}{dt} = A'\cos(\omega t + \phi')$ where $A' = A\omega$ and $\phi' = \phi + \pi/2$. Similarly, the integral of a sinusoid has a new amplitude of A/ω and an additional phase shift of -90 degrees. So, if $Vin = A\sin(\omega t + \phi)$, then

$$\int Vin\,dt = A'\sin(\omega t + \phi') \text{ where } A' = \frac{A}{\omega} \text{ and } \phi' = \phi - \pi/2.$$

If we can somehow model our input and output waves so that we are concerning ourselves just with the changes in amplitude and phase brought about by the integrals and derivatives involved, then we can eliminate the need to use differential equations. Since the phase is an angle, it seems logical to begin with polar coordinates.

B.2 Polar Coordinates
Polar coordinates are a simple re-mapping of the traditional rectangular (x-y) coordinates on a plane into another set of variables, r and θ, where r is the distance from the origin and θ is the angle measured from the positive x axis in counter-clockwise direction. This is shown in figure 4.

To convert a point between the polar and x-y coordinate systems, trigonometric identities must be used. The distance to the origin can be found using the Pythagorean theorem: $r = \sqrt{x^2 + y^2}$ and the angle from the x axis is simply found by $\tan\theta = \dfrac{y}{x}$ or $\theta = \tan^{-1}\left[\dfrac{y}{x}\right]$. To go the other way, we can use $x = r\cos\theta$ and $y = r\sin\theta$.

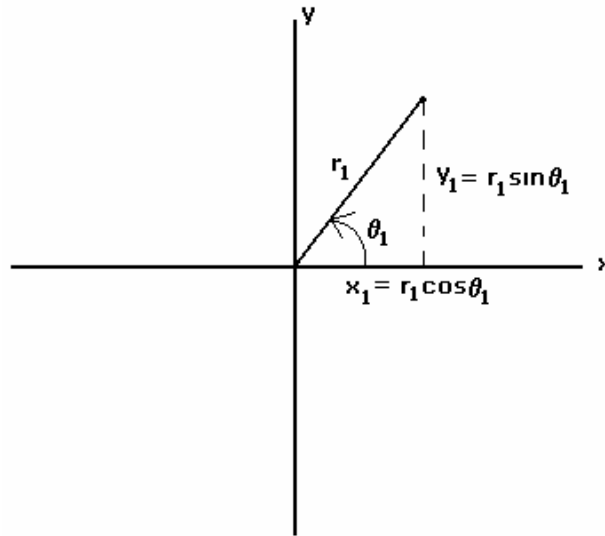Figure 4

If we think of our signal as having a phase offset of ϕ and an amplitude of A, we can model a sinusoid (of angular frequency, ω) as pictured in figure 5. Note that the sinusoid begins at ϕ and cycles around the origin one time for each cycle (ωt). The ray has a length of A, the amplitude of the sinusoid. If we want to get back the original sinusoid, we have only look at either the x coordinate, $x = A \cos(\omega t + \phi)$, or the y coordinate, $y = A\sin(\omega t + \phi)$. Note that ωt represents a complete cycle around the circle (2π) at a rate of ω radians per second. This corresponds to the constant cycling of the sinusoid in time. We could just as easily drop the ωt, since it is a simple multiple of 2π, and we would have an x coordinate, $x = A \cos(\phi)$, and a y coordinate, $y = A \sin(\phi)$.
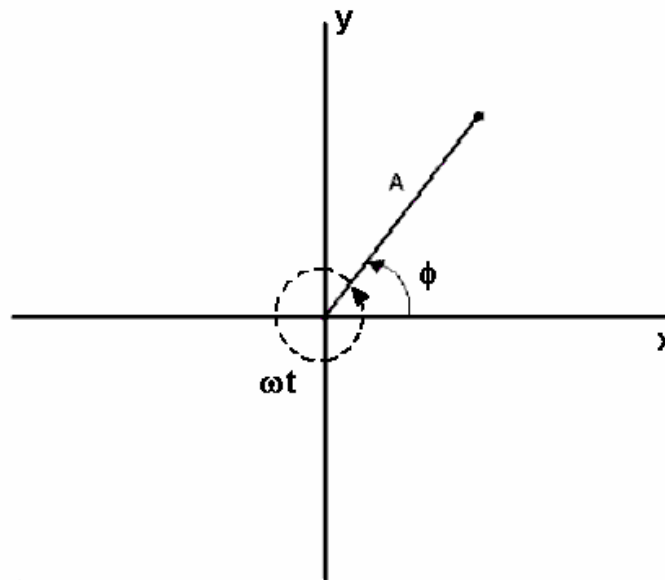

Figure 5

Polar coordinates give us a good way to express the important features of our sinusoid, but we still need a way to operate on the sinusoids to simulate the behavior of the circuit. For this, we will need to rethink our model in terms of complex numbers.

B.3 Complex Numbers
Complex numbers are numbers expressed as ordered pairs, consisting of a "real" part and an "imaginary" part. In other words, a complex number is a point on a plane called the complex plane. A complex number z can be written as $z = x + j\,y$. In a complex number: x is called the "real" part, y is called the "imaginary" part, and $j = \sqrt{-1}$. (Note that in common mathematical notation, i is used for $\sqrt{-1}$, however, in electrical engineering, since i is commonly used for current, we denote $\sqrt{-1}$ with j.) The term "imaginary" comes from the fact that $j = \sqrt{-1}$ is not a "real" number. Complex numbers are generally represented on a plane where y is plotted against x. The x coordinate consists of the basic number line containing all real numbers. The y coordinate is the same set of numbers multiplied by $\sqrt{-1}$. Each complex number, z, is a unique point on the complex plane, shown in figure 6, where the x coordinate is the real part and the y coordinate is the imaginary part.
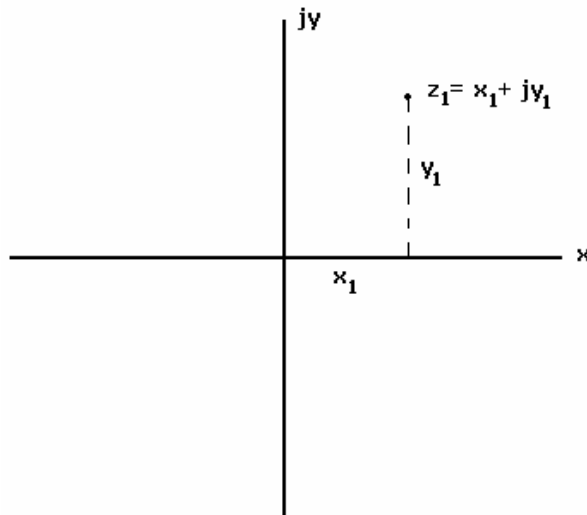

Figure 6

Before we continue there are two useful identities to remember about j:

$$j^2 = j \cdot j = \sqrt{-1} \cdot \sqrt{-1} = -1 \qquad j^2 = -1$$

$$\text{and} \qquad \frac{1}{j} = \frac{1 \cdot j}{j \cdot j} = \frac{j}{-1} = -j \qquad \frac{1}{j} = -j \,.$$

Why do we need complex numbers? What good could having a signal defined in terms of something that doesn't exist possibly do? What we are working towards is defining our sinusoid as a combination of a real part, $x = Acos(\omega t + \phi)$, and an imaginary part, $y = jAsin(\omega t + \phi)$. This will enable us to work easily with sinusoids that have different phases relative to one another. To do this, we need to define complex polar coordinates.

B.4 Complex Numbers in Polar Format

As we mentioned above, points on a plane can be represented in polar coordinates. Similarly any complex number can be written in the polar format shown in figure 7.
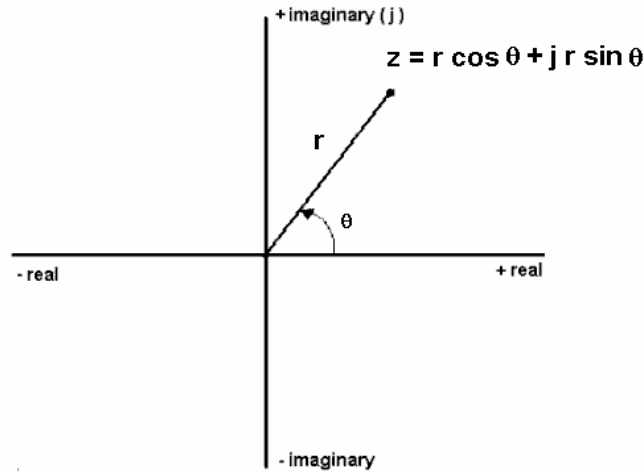

Figure 7

Now we have a complex number, z, defined in terms of a real part, $x = rcos\theta$, and an imaginary part, $y = j\,rsin\theta$. It is a simple matter to translate this so that it represents the phase and amplitude of our sinusoid, shown in figure 8. Note that the constant cycling of ωt (2π for each cycle) does not really affect the value of x and y in the model.
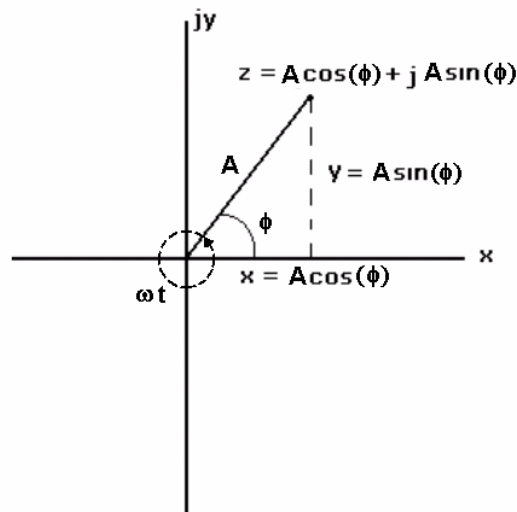

Figure 8

In this representation, we can translate from the time domain to the frequency domain by using $x = A\cos\phi$ and $y = A\sin\phi$ and back to the time domain using $A = \sqrt{x^2 + y^2}$ and $\phi = \tan^{-1}\left[\dfrac{y}{x}\right]$. This is the final representation we will use. We will call this model a phasor and base steady state analysis upon this representation. Before, we go into details, however, let's look at a brief example. Hopefully this will help clarify how the representation works.

B.5 An Example – Adding Two Cosine Waves
We know that a complex polar number is given by z = Acosϕ + j Asinϕ . The whole signal (assuming it is a cosine), can be represented only by the x part of this complex polar number. What will we use y for? Let's look at the example in figure 9.
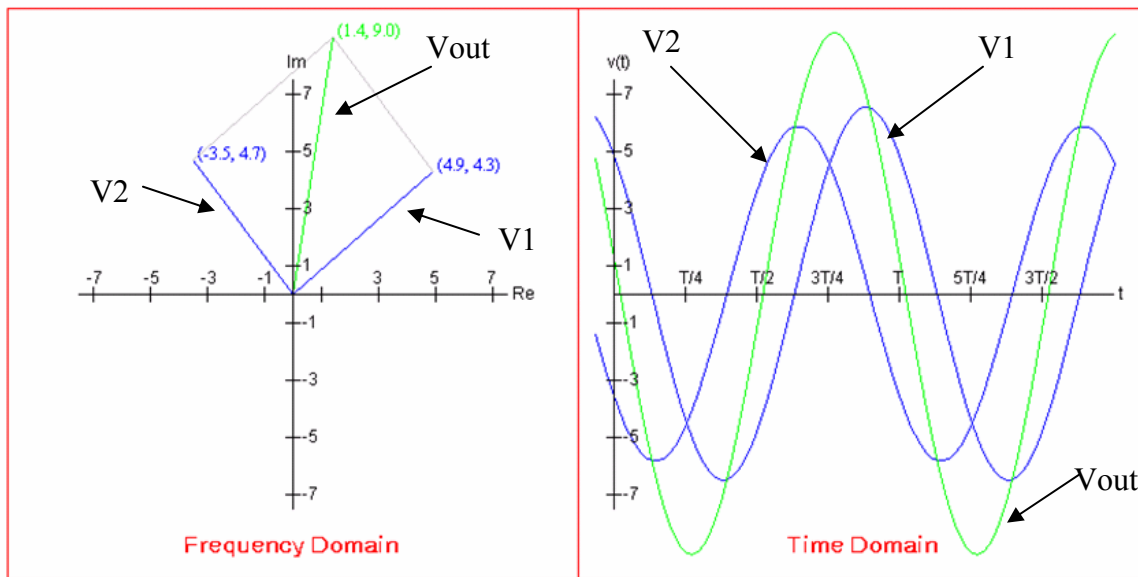

Figure 9

Figure 9 shows two sinusoids in the time domain that we want to add together. The sum (with the largest amplitude) is also shown.
The first input sinusoid is V1 = 6.5 cos(ωt + 0.2π)
and the second sinusoid is V2 = 6 cos(ωt + 0.7π).

If we use complex polar notation (adding the imaginary part), these two signals become:
V1 = 6.5 cos(ωt + 0.2π) + j 6.5 sin(ωt + 0.2π) and
V2 = 6 cos(ωt + 0.7π) + j 6 sin(ωt + 0.7π).

To translate into the frequency domain, we use $x = A\cos\phi$ and $y = A\sin\phi$. This gives us the components of the two input vectors: z1 = 4.9 +j4.3 and z2 = -3.5 + j4.7. We can add the x and y coordinates of these vectors to get the vector sum, z_{out} = 1.4 + j 9.0. This represents the sum of the sinusoids. The real part can be found by determining the

amplitude using $A = \sqrt{x^2 + y^2} = \sqrt{1.4^2 + 9.0^2} = 9.1$ and the phase angle by using

$\phi = \tan^{-1}\left[\dfrac{y}{x}\right] = \tan^{-1}\left[\dfrac{9.0}{1.4}\right] = 0.45\pi \; rad$. This means that the sum is

Vout = 9.1 cos (ωt + 0.45π). Since the amplitude and phase (along with the original frequency) define the new signal, the vectors have given us the sum. Although the imaginary part is inserted at the beginning and discarded at the end, it is instrumental in determining how much of the two signals overlap in time. This representation is even more useful when dealing with multiplication and transfer functions.

C. Phasors

A phasor is a representation of a sinusoidal wave in complex polar form. The variables are expressed in terms of the phase offset and amplitude. Therefore, if we have a sinusoidal signal,

$$v(t) = A\cos(\omega t + \phi),$$

the phasor of this signal would be expressed as

$$\vec{V} = A\cos(\omega t + \phi) + jA\sin(\omega t + \phi).$$

Since we know that the ωt will simply return a multiple of 2π, this can be further simplified to:

$$\vec{V} = A\cos\phi + jA\sin\phi \quad \text{[equation 2]}$$

Phasors have special terminology. The amplitude of the sine wave, is called the *magnitude* of the phasor and is denoted $\left|\vec{V}\right|$. The phase offset of the sine wave is called the phase of the phasor and is denoted $\angle\vec{V}$. The magnitude and phase of a phasor given in terms of Cartesian coordinates, $z = x + j\,y$, can be determined using the equations for polar coordinates:

$$\left|\vec{V}\right| = \sqrt{x^2 + y^2} \quad \text{[equation 3]}$$

$$\text{and} \quad \angle\vec{V} = \tan^{-1}\left(\dfrac{y}{x}\right) \quad \text{[equation 4]}$$

C.1 Euler's Formula
In order to make it easier to manipulate our phasors, we must understand Euler's formula. Euler's formula relates the complex polar definition we have defined to the natural logarithm, e. Euler's formula can be stated as follows:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This means that our definition of phasors can be extended as follows:

$$\vec{V} = Ae^{j\phi} = A\cos\phi + jA\sin\phi.$$

Along with Euler's formula, come several shortcuts that we can use to manipulate phasors. By a natural extension of Euler's formula, it can be shown that, if we have two complex polar numbers, $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then, the product of these two numbers is given by

$$z_3 = z_1 z_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1+\theta_2)},$$

and the quotient of the two numbers is given by

$$z_3 = \frac{z_1}{z_2} = \frac{r_1\ e^{j\theta_1}}{r_2\ e^{j\theta_2}} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}.$$

We can use these rules to multiply phasors:

$$\textit{If } \vec{V}_1 = A_1 e^{j\phi_1} \textit{ and } \vec{V}_2 = A_2 e^{j\phi_2}, \textit{ then } \vec{V}_3 = \vec{V}_1\vec{V}_2 = A_1 A_2 e^{j(\phi_1+\phi_2)}$$

We find that the magnitude of the product is the product of the magnitudes, $\left|\vec{V}_3\right| = \left|\vec{V}_1\right|\left|\vec{V}_2\right| = A_1 A_2$, and the phase of the product is the sum of the phases, $\angle\vec{V}_3 = \angle\vec{V}_1 + \angle\vec{V}_2 = \phi_1 + \phi_2$. We can also use these rules to divide phasors:

$$\textit{If } \vec{V}_1 = A_1 e^{j\phi_1} \textit{ and } \vec{V}_2 = A_2 e^{j\phi_2}, \textit{ then } \vec{V}_3 = \frac{\vec{V}_1}{\vec{V}_2} = \frac{A_1}{A_2} e^{j(\phi_1-\phi_2)}$$

We find that the magnitude of the quotient is the quotient of the magnitudes, $\left|\vec{V}_3\right| = \dfrac{\left|\vec{V}_1\right|}{\left|\vec{V}_2\right|} = \dfrac{A_1}{A_2}$ and the phase of the quotient is the difference of the phases, $\angle\vec{V}_3 = \angle\vec{V}_1 - \angle\vec{V}_2 = \phi_1 - \phi_2$.

If the two phasors are expressed as a ratio of two Cartesian complex numbers,

$$\vec{V}_1 = x_1 + jy_1 \quad and \quad \vec{V}_2 = x_2 + jy_2 \quad and \quad \vec{V}_3 = \frac{\vec{V}_1}{\vec{V}_2} = \frac{x_1 + jy_1}{x_2 + jy_2},$$

then the magnitude and phase of the quotient can be found as follows:

$$\left| \vec{V}_3 \right| = \frac{\left| \vec{V}_1 \right|}{\left| \vec{V}_2 \right|} = \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} \quad \text{[equation 5]}$$

and

$$\angle \vec{V}_3 = \angle \vec{V}_1 - \angle \vec{V}_2 = \tan^{-1}\left(\frac{y_1}{x_1} \right) - \tan^{-1}\left(\frac{y_2}{x_2} \right). \quad \text{[equation 6]}$$

C.2 Phasors and Transfer Functions

Recall that we are looking for a transfer function that relates the output to the input signal of a circuit using the relationship in equation 1. We have an input signal of arbitrary amplitude and phase, and a corresponding output signal,

$$v_{in}(t) = A_{in} \cos(\omega t + \phi_{in}) \; and \; v_{out}(t) = A_{out} \cos(\omega t + \phi_{out})$$

By defining the input and output in terms of phasors,

$$\vec{V}_{in} = A_{in} e^{i(\omega t + \phi_{in})} \; and \; \vec{V}_{out} = A_{out} e^{i(\omega t + \phi_{out})}$$

 we can express the function, H, as follows:

$$\vec{H} = \frac{\vec{V}_{out}}{\vec{V}_{in}} = \frac{A_{out} e^{i(\omega t + \phi_{out})}}{A_{in} e^{i(\omega t + \phi_{in})}} = \frac{A_{out} e^{j\omega t} e^{j\phi_{out}}}{A_{in} e^{j\omega t} e^{j\phi_{in}}} = \frac{A_{out} e^{j\phi_{out}}}{A_{in} e^{j\phi_{in}}} = \frac{A_{out}}{A_{in}} e^{j(\phi_{out} - \phi_{in})}$$

If we have the phasor for the input and a complex polar expression for H, we can calculate the phasor for the output as follows:

$$\vec{V}_{out} = \vec{H} \cdot \vec{V}_{in} \quad \text{[equation 7]}$$

By defining H, we have also defined a new type of phasor. The magnitude of H, is not the amplitude of any sinusoidal signal, it is a factor that we multiply by the input amplitude to determine the output amplitude. Likewise, the phase of H is not a phase angle, but the change in phase between the input and the output wave. Thus, given the transfer function, H, the amplitude and phase of the output phasor can be determined by

$$A_{out} = \left| \vec{H} \right| \cdot A_{in} \quad \text{[equation 8]}$$

and

$$\phi_{out} = \angle \vec{H} + \phi_{in} \quad \text{[equation 9]}$$

Now we need to find a way to use the components in the circuit itself to derive some phasor, H, that allows us to map the input phasor of the circuit to an appropriate output phasor.

D. Finding Transfer Functions

We know that by using phasors and complex polar coordinates, we can greatly simplify the problem of determining the output of an AC circuit for any input. It still remains, however, to develop a strategy for finding the transfer function phasor.

D.1 The Influence of Resistors, Capacitors, and Inductors
Conceptually, H, defines how the circuit alters the phase and amplitude of the input wave to produce the output wave. Each component in the circuit is influencing the amplitude and the phase in some way. We can use the equations that govern the behavior of each of the three basic passive components (resistors, capacitors, and inductors) to determine the nature of the influence of each.

In order to do this, we will consider a sinusoidal input, $v_{in}(t) = A_{in} \sin(\omega t + \phi_{in})$, and examine the way in which each device alters its phase and amplitude. Then we will examine what this influence means in the complex polar coordinate system. We will define the influence of each component in terms of a complex quantity called *impedance* and denote it by the letter Z.

The fundamental equation that governs the behavior of a resistor is $v_R(t) = i_R(t) \cdot R$. If we let $v_R(t) = A_{in} \sin(\omega t + \phi_{in})$, then the output signal will be $v_{out}(t) = (R \cdot A_{in}) \sin(\omega t + \phi_{in})$. Note that the resistor changes the amplitude of the input by a factor of R, but it does not affect the phase at all. Since $A_{out} = \left| \vec{H} \right| \cdot A_{in}$ and $\phi_{out} = \angle \vec{H} + \phi_{in}$, the magnitude of a phasor showing the influence of a resistor, Z_R, must be R and the phase must be zero. In complex polar form, this would mean $\vec{Z}_R = \text{Re}^{j(0)} = R \cos 0 + jR \sin 0 = R$.
Therefore, the impedance of a resistor, is given simply by

$$\vec{Z}_R = R \quad \text{[equation 10]}$$

The fundamental equation that governs the behavior of an inductor is $v_L(t) = L\dfrac{di_L(t)}{dt}$.

If we let $v_L(t) = A_{in}\sin(\omega t + \phi_{in})$, then the output signal will be

$v_{out}(t) = (L \cdot A_{in} \cdot \omega)[\cos(\omega t + \phi_{in})] = \omega L A_{in}\sin(\omega t + \phi_{in} + \pi/2)$. The inductor changes the amplitude of the input by a factor of ωL and shifts the phase by +π/2. (Recall that the return from sine to cosine shifts the phase by +π/2.) Since

$A_{out} = |\vec{H}| \cdot A_{in}$ and $\phi_{out} = \angle\vec{H} + \phi_{in}$, the magnitude of a phasor showing the influence of a inductor, Z_L, must be ωL and the phase must be +π/2. In complex polar form, this would mean $\vec{Z}_L = \omega L \cdot e^{j(+\pi/2)} = \omega L \cdot \cos\left(+\pi/2\right) + j\omega L \cdot \sin\left(+\pi/2\right) = j\omega L$.

Therefore, the impedance of a capacitor, is given by

$$\vec{Z}_L = j\omega L \quad \text{[equation 11]}$$

The fundamental equation that governs the behavior of a capacitor is $i_C(t) = C\dfrac{dv_C(t)}{dt}$.

If we solve this for the voltage, we get that $v_C(t) = \dfrac{1}{C}\int i_C(t)\,dt$. If we let

$v_C(t) = A_{in}\sin(\omega t + \phi_{in})$, then the output signal will be

$v_{out}(t) = \left(\dfrac{1}{C} \cdot A_{in} \cdot \dfrac{1}{\omega}\right)[-\cos(\omega t + \phi_{in})] = \left(\dfrac{A_{in}}{\omega C}\right)\sin(\omega t + \phi_{in} - \pi/2)$. The capacitor changes the amplitude of the input by a factor of 1/ωC and shifts the phase by -π/2. (The return from cosine to sine shifts the phase by +π/2 and the negative sign shifts it by -π. The net phase shift caused by the integration is -π/2.) Since

$A_{out} = |\vec{H}| \cdot A_{in}$ and $\phi_{out} = \angle\vec{H} + \phi_{in}$, the magnitude of a phasor showing the influence of a capacitor, Z_C, must be 1/ωC and the phase must be -π/2. In complex polar form, this would mean

$\vec{Z}_C = \left(\dfrac{1}{\omega C}\right)e^{j(-\pi/2)} = \left(\dfrac{1}{\omega C}\right)\cos\left(-\pi/2\right) + j\left(\dfrac{1}{\omega C}\right)\sin\left(-\pi/2\right) = -j\left(\dfrac{1}{\omega C}\right) = \dfrac{1}{j\omega C}$.

Therefore, the impedance of a capacitor, is given by

$$\vec{Z}_C = \dfrac{1}{j\omega C} \quad \text{[equation 12]}$$

We can now use these complex expressions to define transfer functions of RLC circuits.

D.2 Transfer Functions and Complex Impedance
Recall that, in figure 2, we used a voltage divider to find the transfer function of a resistive circuit. We can extend the use of voltage dividers to complex phasors in order to find transfer functions for passive circuits containing capacitors, inductors and resistors. In figure 10, we have taken the circuit in figure 3 and represented the influence of each component by its complex impedance. To this, we can apply the voltage divider rule in the same way as we did in figure 2 to find H.
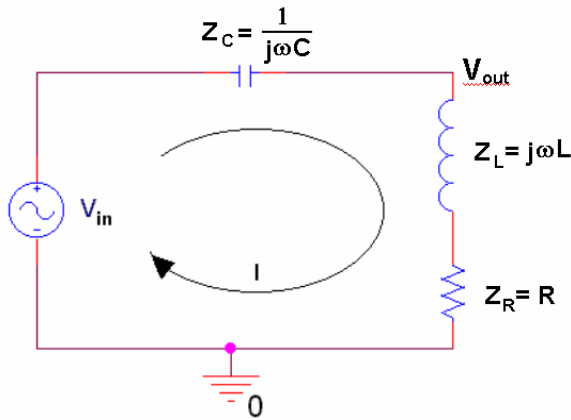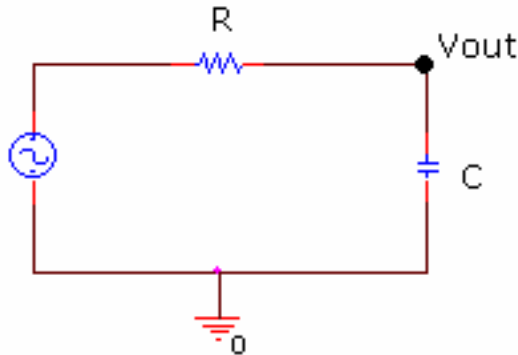

$$V_{out} = V_{in} * \frac{Z_L + Z_R}{Z_C + Z_L + Z_R}$$

$$H = \frac{V_{out}}{V_{in}} = \frac{Z_L + Z_R}{Z_C + Z_L + Z_R}$$

Figure 10

Let's look at some simpler examples.

D.3 An RC Circuit Example
Let's see how transfer functions work by considering the following circuit:

First note the impedances of the two devices: $Z_R = R \quad and \quad Z_c = \dfrac{1}{j\omega C}$. Using the

voltage divider rule, we can write the output as $\vec{V}_{out} = \dfrac{Z_C}{Z_R + Z_C} \vec{V}_{in}$ or equivalently

$$H(j\omega) = \frac{\vec{V}_{out}}{\vec{V}_{in}} = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}}.$$ To make working with the transfer function easier, it is

best to simplify it by multiplying by $\dfrac{j\omega C}{j\omega C}$:

$$H(j\omega) = \frac{\vec{V}_{out}}{\vec{V}_{in}} = \frac{\dfrac{1}{j\omega C} \cdot j\omega C}{\left(R + \dfrac{1}{j\omega C}\right) \cdot j\omega C} = \frac{1}{j\omega RC + 1}$$

Therefore, the transfer function of the circuit is

$$H(j\omega) = \frac{1}{1 + j\omega RC}$$

Now, using equation 5, we can find an expression for the magnitude:

$$\left| H(j\omega) \right| = \frac{\left| 1 + j0 \right|}{\left| 1 + j\omega RC \right|} = \frac{\sqrt{1^2 + 0^2}}{\sqrt{1^2 + (\omega RC)^2}} = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

and, using equation 6, the phase:

$$\angle H(j\omega) = \angle(1 + j0) - \angle(1 + j\omega RC) = \tan^{-1}\left(\frac{0}{1}\right) - \tan^{-1}\left(\frac{\omega RC}{1}\right) = -\tan^{-1}(\omega RC).$$

What is the output of this circuit, if we let R=1KΩ and C=1μF, and if the input is $V_{in}(t) = A_{in}cos(\omega t + \phi_{in}) = 2Vcos(2\pi K\, t + \pi/4)$?

We know from the input signal that $\omega = 2\pi K$ rad/sec. We can substitute the angular frequency and the component values into the equations for the magnitude and phase of $H(j\omega)$ to find:

$$\left| H(j2K\pi) \right| = \frac{1}{\sqrt{1 + (2\pi K1K(1\mu))^2}} = 0.157 \quad \text{and} \quad \angle H(j\omega) = -\tan^{-1}(2\pi K1K1\mu) = -1.41$$

We can now find the amplitude, phase, and function for the output of this circuit at $2\pi K$ rad/sec using equations 8 and 9.
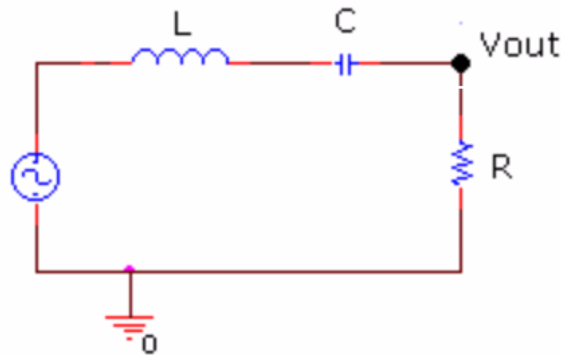
$A_{out} = |H(j\omega)|A_{in} = (0.157)(2V) = 0.314V$

$\phi_{out} = \angle H(j\omega) + \phi_{in} = 0.785\text{-}1.41 = -0.625 \text{ rad}$

Therefore the output of the circuit for this input is

$V_{out}(t) = 0.314V \cos(2\pi K\, t - 0.625)$

D.4 An RLC Circuit Example
Let's look at an example with all three types of components.

First we will find the transfer function of this circuit:

$$H(j\omega) = \frac{R}{j\omega L + \dfrac{1}{j\omega C} + R} = \frac{j\omega RC}{j^2\omega^2 LC + 1 + j\omega RC} = \frac{j\omega RC}{(1 - \omega^2 LC) + j\omega RC}.$$

Since this is a more complex example, let's substitute in the numerical values before we
find expressions for the magnitude and the phase. If we let L=1mH, C=1µF, and R=1KΩ
and we use the same input in the previous example,
$V_{in}(t) = A_{in}\cos(\omega t + \phi_{in}) = 2V\cos(2\pi K\, t + \pi/4)$.

The transfer function becomes:

$$H(j\omega) = \frac{j(2\pi K)(1K)(1\mu)}{(1 - (2\pi K)^2(1m)(1\mu)) + j(2\pi K)(1K)(1\mu)} = \frac{j6.3}{(1 - 39.5m) + j6.3} = \frac{j6.3}{0.96 + j6.3}$$

Now we can find the magnitude and phase using equations 5 and 6:

$$|H(j\omega)| = \frac{6.3}{\sqrt{(0.96)^2 + (6.3)^2}} = 1.0$$

$$\text{and } \angle H(j\omega) = \tan^{-1}\left(\frac{6.3}{0}\right) - \tan^{-1}\left(\frac{6.3}{0.96}\right) = \pi/2 - 1.42 = 0.15\,rad$$

This means that the output of this circuit for the given input is

$V_{out}(t) = 2(1.0)V\cos(2\pi K\,t + (.785 + 0.15)) = 2V\cos(2\pi K\,t + .935)$

Note that the phase of j6.3 is π/2. This is because it is a positive imaginary number and therefore lies on the +j axis (in figure 8) which is 90 degrees from the positive x axis. For details on finding phases, consult appendix A.

E. Filters

Filters are a fundamental concept in electronics. Many RL, RC and RLC circuits act as some type of filter. Figure 11 shows the transfer functions of the four basic filter types: low pass filter, high pass filter, band pass filter, and band reject filter.

low pass filter   high pass filter   band pass filter   band reject filter
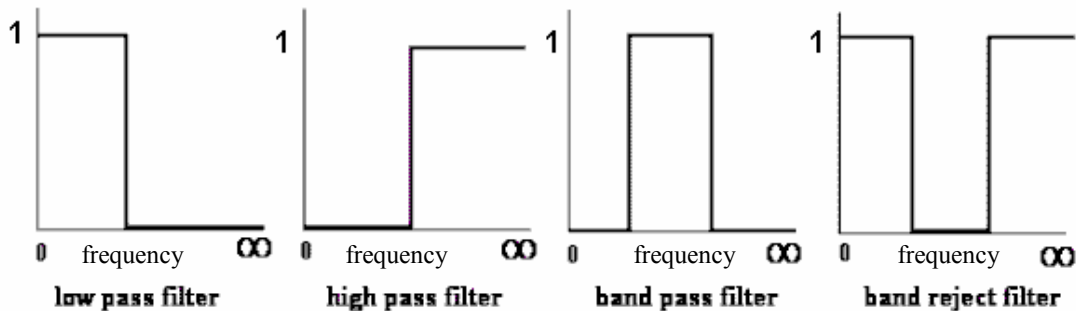
The names of the filters indicate exactly what they do. A low pass filter, for example, passes low frequencies. This means that if the input has a low frequency, the output will be the same as the input. (Note that the value of the transfer function of a low pass filter at low frequencies is 1.) If the input has a high frequency, the output will be multiplied by the value of the transfer function at high frequencies, which is zero. Hence, the output for a high frequency input will be filtered out. The filter has passed low frequencies and rejected high frequencies.

In order to figure out what type of filter a circuit is, it is helpful to know how it behaves at very high and very low frequencies. The transfer function can be used to easily determine both the magnitude and phase of the output of a circuit at these extremes.

E.1 Transfer Functions at Low Frequencies
First, let's consider how to find the magnitude and phase of a transfer function at low frequencies. If we have a transfer function with a general form:

$$H(j\omega) = \frac{a_2\omega^2 + a_1\omega + a_0}{b_2\omega^2 + b_1\omega + b_0}$$, then we first find how this function looks at very low

frequencies. To do this you must find the single term in the numerator and the denominator which dominates the behavior of the function as ω gets very small. This term will be the one with the smallest power of ω. (Remembering, of course, that $\omega^0 = 1$.)

To understand why this works, consider $\omega = 10^{-3}$. Since $\omega^2 = 10^{-6}$, $\omega^1 = 10^{-3}$, and $\omega^0 = 1$, $a_2\omega^2 + a_1\omega + a_0 = a_2 \times 10^{-6} + a_1 \times 10^{-3} + a_0$ which is approximately a_0. If you assume all the a_i are 1, then the term becomes 1.001001, which rounds to 1. As ω gets even smaller, the number just gets closer to 1. The same holds true for the denominator. For more on taking limits, see Appendix B.

In our first example from the last section, $H(j\omega) = \dfrac{1}{1 + j\omega RC}$, the lowest power of ω in

both the numerator and denominator is ω^0 (or 1) and the function simplifies to:

$$H_{LOW}(j\omega) = \frac{1}{1} = 1$$ (This is the complex number "1 + j 0").

We can use equations 5 and 6 to find the magnitude and phase.

$$\left|H_{LOW}(j\omega)\right| as\, \omega \to 0 = \left|1\right| = 1 \qquad \angle H_{LOW}(j\omega) = \tan^{-1}\left(\frac{0}{1}\right) = 0 \;\; rad$$

Note that when we calculate the magnitude at low frequencies, we want to take the magnitude of $H_{LOW}(j\omega)$ and then take its limit as ω approaches 0. The phase should be found by applying the phase equation to $H_{LOW}(j\omega)$, NOT to its magnitude. Taking the phase of the magnitude is meaningless.

Let's look at our other example, $H(j\omega) = \dfrac{j\omega RC}{(1 - \omega^2 LC) + j\omega\, RC}$. The term with the

lowest power of ω in the numerator is jωRC. The term with the lowest power of ω in the denominator is 1. Therefore, the transfer function at low frequencies is:

$$H_{LOW}(j\omega) = \frac{j\omega RC}{1} = j\omega RC$$ (This is complex number "0 + jωRC")

To find the magnitude, we take the limit of the magnitude of this function as ω approaches zero.

$$\left| H_{LOW}(j\omega) \right| as\, \omega \to 0 = \omega RC\, as\, \omega \to 0 = 0$$

To find the phase, go back to the general function for $H_{LOW}(j\omega)$ and find its phase.

$$\angle H_{LOW}(j\omega) = \tan^{-1}\left(\frac{\omega RC}{0}\right) = \pi/2\ rad \quad why?$$

In this particular case, we have run into a problem with the phase equation. We are dealing with a point where the tangent approaches infinity. However, since $H_{LOW}(j\omega)$, $j\omega RC$, is a positive imaginary number, we know its phase must be $\pi/2$. The special cases (when either the real or imaginary term is zero) are fairly easy to deal with. Decide whether the complex number you found for $H_{LOW}(j\omega)$ is positive real, negative real, positive imaginary, or negative imaginary. Then, use the following chart:

Case	$\tan^{-1}(x+jy)$ in radians
(+real)	0
(+imaginary)	$\pi/2$
(-real)	$-\pi$ or π
(-imaginary)	$-\pi/2$

Although finding the phase for the limits of transfer functions is fairly straight forward (because the limit always ends up one of the four cases in the chart), finding phases in the general case can be quite complicated. Please see appendix A for more information.

E.2 Transfer Functions at High Frequencies
For high frequencies, we must consider what happens when ω gets very large. Let us use the general form again and let ω be 10^{+3}. This means that $\omega^2 = 10^{+6}$, $\omega^1 = 10^{+3}$, and $\omega^0 = 1$ and that the numerator is $a_2 \times 10^{+6} + a_1 \times 10^{+3} + a_0$. If you assume all the a_i are 1, than the term becomes 1,001,001, which rounds to 1 million. As ω gets even larger, the number just gets closer to a_2. Clearly, the contribution of a_2 is much more than the others. Therefore, at high frequencies, the dominant term in both the numerator and denominator has the highest order of ω. For additional information on taking limits, see appendix B.

In our first example, $H(j\omega) = \dfrac{1}{1 + j\omega RC}$, the numerator has only one term, so the highest power of ω is 0. In the denominator, we have two terms and we use the ω^1 term, $j\omega RC$. Therefore, at high frequencies, the function that governs the behavior is

$$H_{HIGH}(j\omega) = \frac{1}{j\omega RC}. \quad \text{(This is imaginary number "}0 - j\,\omega RC\text{" because } 1/j = -j.)$$

Now we can find the magnitude and phase.

$$\left| H_{HIGH}(j\omega) \right| as\ \omega \to \infty = \frac{1}{\omega RC}\ as\ \omega \to \infty = \frac{1}{\infty} = 0 \qquad \angle H_{HIGH}(j\omega) = "-imaginary" = -\frac{\pi}{2}\ rad$$

Since the transfer function is a negative imaginary number, the phase (from the chart) must be -π/2. For more information about determining phase, see appendix A.

In our other example, $H(j\omega) = \dfrac{j\omega RC}{(1-\omega^2 LC) + j\omega\,RC}$. At high frequencies, this simplifies to

$$H(j\omega) = \frac{j\omega RC}{-\omega^2 LC} = \frac{-jR}{\omega L}. \quad \text{(This is imaginary number "0 – j (R/\omega L)".)}$$

 The magnitude and phase of this function at high frequencies is

$$\left| H_{HIGH}(j\omega) \right|\ as\ \omega \to \infty = \frac{R}{\omega L}\ as\ \omega \to \infty = 0 \quad and \quad \angle H_{HIGH}(j\omega) = "-imaginary" = -\frac{\pi}{2} rad$$

E.3 Filters and Limits
Once we have the limits of a transfer function, it is simple to determine the type of filter it is. In our first example, $H(j\omega) = \dfrac{1}{1 + j\omega RC}$, the magnitude at low frequencies is 1 and the magnitude at high frequencies is 0. It must be a low pass filter.   At low frequencies, this filter will not change the phase of the signal at all and at high frequencies, there will be a -π/2 phase shift.

Our second example, $H(j\omega) = \dfrac{j\omega RC}{(1-\omega^2 LC) + j\omega\,RC}$, has a magnitude of 0 both at low frequencies and at high frequencies. This does not mean that it is zero at all frequencies. It means that it is a band pass filter. It rejects both very low and very high frequencies, but passes some band of frequencies in between. The phase of this function shifts from +π/2 to -π/2 over the frequency range from zero to infinity.

In order to understand how a filter functions, we must know how it behaves at high and low frequencies, but we must also know what else it does. Where does it transition from high to low?...from low to high? Where is the pass or reject band on a band filter? How wide is the band? To answer these questions, we must know how to find the corner frequency and the resonant frequency.

E.4 The Corner Frequency

We have established that our first example, $H(j\omega) = \dfrac{1}{1 + j\omega RC}$, is a low pass filter. We still don't know, however, the frequency at which the filter switches from passing the

input to rejecting it. This is called the corner frequency. In an ideal filter, the transfer function would switch instantaneously from 1 to 0. In reality, there is a range over which the value of the transfer function goes from 1 to 0. We assign a single frequency to define the approximate location of this area and call it the corner frequency. By definition, the corner frequency is the location at which the value of the transfer function is $\dfrac{1}{\sqrt{2}}$ or about 0.707. This point is chosen since the power that goes through the circuit, at this frequency, is half of the input power.

For our first example we can use the definition of the corner frequency to derive an equation that we can use to find it for any simple RC circuit.

$$H(j\omega) = \frac{1}{1+j\omega RC} \qquad |H(j\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{2}} \qquad \frac{1}{1+(\omega RC)^2} = \frac{1}{2}$$

$$2 = 1+(\omega RC)^2 \qquad \frac{1}{(RC)^2} = \omega^2 \qquad \omega_c = \frac{1}{RC}$$

ω_c is the frequency which defines about where our simple RC low pass filter switches from high to low. As a matter of fact, the corner frequency for any simple RC filter is given by

$$\omega_c = \frac{1}{RC} \quad \text{[equation 13]}$$

You can go through a similar process to prove that the corner frequency for a simple RL circuit is

$$\omega_c = \frac{R}{L} \quad \text{[equation 14]}$$

E.5 The Resonant Frequency
For the circuits that include capacitors and inductors, such as our second example, $H(j\omega) = \dfrac{j\omega RC}{(1-\omega^2 LC)+j\omega RC}$, another special frequency, called the resonant frequency, is defined. We already know that this circuit is a band pass filter. The resonant frequency of a simple RLC circuit like this one, is the frequency at which resonance between the capacitor and the inductor occurs. For the practical purposes of this course, the resonant frequency occurs when the $(1-\omega^2 LC)$ term in the denominator goes to zero. This occurs at:

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad \text{[equation 15]}$$

The above is the definition of resonant frequency we use in this course. It is an approximation to the actual resonant frequency of any simple RLC circuit. In the analysis of more complex circuits, engineers look for poles (locations where the denominator of the transfer function is equal to zero) and zeros (locations where the numerator of the transfer function is equal to zero). We don't have time to go into this amount of detail, so we use this fairly accurate (and very simple) approximation.

In the case of our circuit, the resonant frequency defines the center of the pass band. This filter also has two corners (that define the width of the band). These can be found using the corner equations: $\omega_c = \dfrac{1}{RC}$ and $\omega_c = \dfrac{R}{L}$. RLC filters can be low pass, high pass, band pass or band reject. In the band filters, the resonant frequency gives you the location of the center of the band. In low and high pass filters, it gives you the approximate location of the corner.

F. A More Complicated Example

In this last example, we will examine a band pass filter with some components in parallel. This is the most complex type of transfer function circuit you are likely to have to analyze in this class.
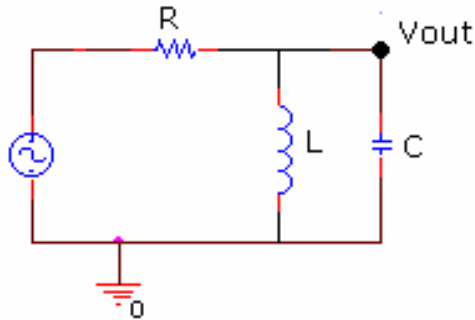
Recall that voltage dividers only work for series circuits. Hence, in order to determine the transfer function for this circuit, we must first combine the two parallel components. This is done using the combination rules for parallel resistors.

$$Z_{CL} = \frac{Z_L \cdot Z_C}{Z_L + Z_C} = \frac{j\omega L \cdot \dfrac{1}{j\omega C}}{j\omega L + \dfrac{1}{j\omega C}} = \frac{j\omega L}{j^2\omega^2 LC + 1} = \frac{j\omega L}{1 - \omega^2 LC}$$

Using the voltage divider rule, the transfer function is given by:

$$H(j\omega) = \frac{Z_{CL}}{R + Z_{CL}} = \frac{\dfrac{j\omega L}{1 - \omega^2 LC}}{R + \dfrac{j\omega L}{1 - \omega^2 LC}} \qquad multiply\ by\ \frac{1 - \omega^2 LC}{1 - \omega^2 LC} \qquad H(j\omega) = \frac{j\omega L}{R(1 - \omega^2 LC) + j\omega L}$$

Now that we have H(jω) we can look at the behavior at low and high frequencies:

$$H_{LOW}(j\omega) = \frac{j\omega L}{R} \qquad \left| H_{LOW}(j\omega) \right| \omega \to 0 = 0 \qquad \angle H_{LOW}(j\omega) = \frac{\pi}{2}\,rad$$

$$H_{HIGH}(j\omega) = \frac{j\omega L}{-\omega^2 LRC} = \frac{-j}{\omega RC} \quad \left| H_{HIGH}(j\omega) \right| \omega \to \infty = \left| \frac{1}{\infty} \right| = 0 \quad \angle H_{HIGH}(j\omega) = -\frac{\pi}{2}\,rad$$

This filter, although it approaches 0 at both high and low frequencies, does not block all frequencies. It will pass a band around the resonant frequency, which by definition is $\omega_0 = \dfrac{1}{\sqrt{LC}}$.

More useful information about a filter can be found by finding the transfer function, magnitude and phase at the resonant frequency. To do this, you simply substitute the expression for the resonant frequency, ω₀, into the equation for H(jω).

$$H(j\omega_0) = \frac{j\omega_0 L}{R(1 - \omega_0^2 LC) + j\omega_0\,L} = \frac{j\left(\dfrac{1}{\sqrt{LC}}\right)L}{R\left(1 - \left(\dfrac{1}{\sqrt{LC}}\right)^2 LC\right) + j\left(\dfrac{1}{\sqrt{LC}}\right)L} \quad = \quad 1$$

$$\left| H(j\omega_0) \right| = 1 \qquad \angle H(j\omega_0) = 0\,rad$$

You can see that this filter does indeed pass a band of frequencies around the resonant frequency, because the magnitude of the transfer function at the resonant frequency is 1. Therefore, for some band of frequencies centered around the resonant frequency, the output will be equal to the input. The width of the band can be found using the corner frequency equations. Further information can be found in the Gingrich on-line notes at http://www.phys.ualberta.ca/~gingrich/phys395/notes/node20.html.

G. Conclusion

Transfer functions relate the output to the input of a circuit. For AC circuits, we can use phasors to easily find and manipulate transfer functions. Phasors are defined in terms of the complex polar coordinate system, because AC signals are sinusoids which are easily

manipulated using this representation. Transfer functions are useful for finding the output of a circuit for any given input. They can also be used to determine if a circuit is behaving as a filter, and enable us to find important features of that filter. Other useful features of filters are the corner frequency and the resonant frequency.

Appendix A – More about Phases

Phases can be determined by looking at the real and imaginary parts of the H(jω) function. The general equation for phase is given by:

$$\angle \vec{V} = \tan^{-1}\left(\frac{y}{x}\right) \qquad when \ \vec{V} = x + j\,y$$

Calculating phases using the inverse tangent function
If the transfer function is given as a ratio of two complex numbers, then the phase is given by the difference between the phases of the numerator and denominator:

$$if \quad H(j\omega) = \frac{x_1 + j\,y_1}{x_2 + j\,y_2} \quad then \quad \angle H = \tan^{-1}\left(\frac{y_1}{x_1}\right) - \tan^{-1}\left(\frac{y_2}{x_2}\right)$$

If x_1, y_1, x_2 and y_2 are all positive, then the phase changes are all in the first quadrant, and the equation can be applied directly with a calculator. If one or more of them is negative, then one must worry about which quadrant the phase angle is in. The most reliable way to determine a phase change is to take the absolute value of the x and y coordinates of a complex number, calculate $\tan^{-1}(|y/x|)$ to find the reference angle, use the signs of x and y to determine the quadrant, and find the phase based on the reference angle and the quadrant.
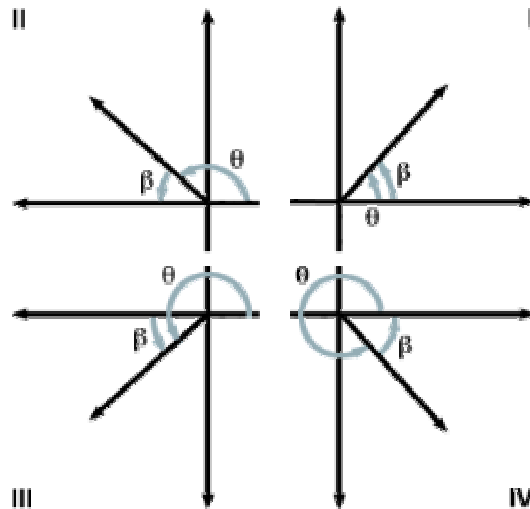
In the figure above, β is the reference angle for θ. We want to find θ -- the actual phase.
tan⁻¹(|y/x|) will always give us the reference angle β. We can find θ based on the sign and
the quadrant:

x	y	quadrant	θ (radians)	θ (degrees)
x>0	y>0	I	β	β
x<0	y>0	II	π - β	180 - β
x<0	y<0	III	β - π	β - 180
x>0	y<0	IV	-β	-β

Note that all angles in the above chart represent a phase shift between -π and +π radians
(between -180 and +180 degrees).

some examples: $H(j\omega) = \dfrac{3 + j4}{4 + j3}$

    numerator: β = tan⁻¹(|4/3|) = 0.93 (x>0, y>0, Q1)
      ∠num = 0.93
    denominator: β = tan⁻¹(|3/4|) = 0.54 (x>0, y>0, Q1)
      ∠den = 0.54
   ∠ H = ∠num - ∠den = 0.93-0.54 = 0.39 rad

$H(j\omega) = \dfrac{-3 + j4}{4 - j3}$

    numerator: β = tan⁻¹(|4/3|) = 0.93 (x<0, y>0, Q2)
      ∠num = 3.14-0.93 = 2.21
    denominator: β = tan⁻¹(|3/4|) = 0.54 (x>0, y<0, Q3)
      ∠den = 0.54-3.14 = -2.60
   ∠ H = ∠num - ∠den = 2.21-(-2.60) = 4.81 rad = -1.47 rad

$H(j\omega) = \dfrac{-3 - j4}{4 + j3}$

    numerator: β = tan⁻¹(|4/3|) = 0.93 (x<0, y<0, Q4)
      ∠num = -0.93
    denominator: β = tan⁻¹(|3/4|) = 0.54 (x>0, y>0,Q1)
      ∠den = 0.54
   ∠ H = ∠num - ∠den = -0.93-0.54 = -1.47 rad

Special cases for finding phases
Note that a calculator can be used to find phases where the real and imaginary parts of a
complex number have a non-zero value. What does one do when the real or imaginary
part of a complex number is zero? These cases are best determined by examination using
the complex plane. Most functions we deal with in this class can be found using this
simple method, which avoids the use of the calculator entirely.

Whenever one is considering the limits of the transfer function at very high or very low frequencies, there will be only one term in the numerator and one term in the denominator of the simplified transfer function. For these two types of limits, either the real or imaginary part of the complex transfer function will always be zero. This makes finding the phase with a calculator difficult. It is easier to find the phase in these common cases by using the complex plane shown below. If we determine which of the four axes the function lies on, we can determine its phase by inspection:
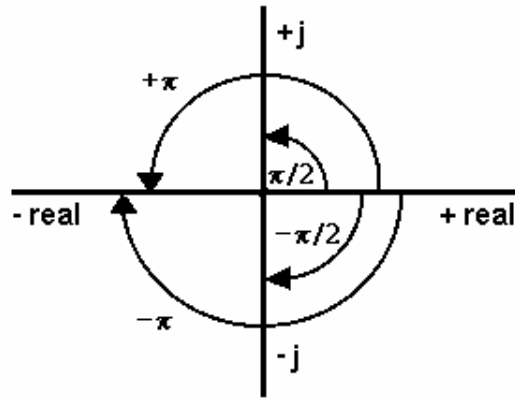
In the above figure, any point on the axis listed has the indicated phase:

+real (ex. R) → 0 radians
+j (ex. jωL) → π/2 radians
-real (ex. -ω^{2}LC) → π (or -π) radians
-j (ex. 1/jωC or -jωL) → -π/2 radians

Or one can remember the simple relationships in the following table:

Case	real part (x)	Imaginary part (y)	tan^{-1}(x+jy) in degrees	tan^{-1}(x+jy) in radians
(b) (+real)	X	0	0	0
(c) (+j)	0	Y	90	π/2
(d) (-real)	-X	0	-180 or 180	-π or π
(e) (-j)	0	-Y	-90	-π/2

Another common case, is when the absolute value of the real and imaginary parts of the complex number are equal. These cases are easily determined using a calculator or by inspection because the transfer function lies along either y=x or y=-x.
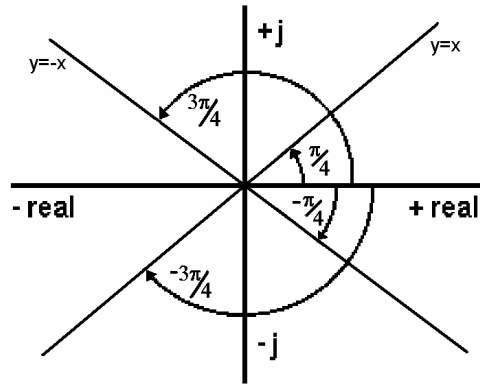
In the above figure, any point on y=x or y=-x when |y|=|x| has the indicated phase:

      x>0 and y>0 (ex. 1+j) → π/4 radians
      x<0 and y>0 (ex. -1+j) → 3π/4 radians
      x<0 and y<0 (ex. -1-j) → -3π/4 radians
      x>0 and y<0 (ex. 1-j) → -π/4 radians

The following chart contains some other useful values of $\tan^{-1}$. Note that these values can be determined simply by finding the quadrant in the real/imaginary plane.

Case	real part (x)	Imaginary part (y)	$\tan^{-1}(x+jy)$ in degrees	$\tan^{-1}(x+jy)$ in radians
(a)	0	0	0	0
(f) A(1+j)	A	A	45	π/4
(g) A(-1-j)	-A	-A	-135	-3π/4
(h) A(1-j)	A	-A	-45	-π/4
(i) A(-1+j)	-A	A	135	3π/4

Some examples: $H(j\omega) = R + j\omega L / (R + j\omega L + 1/j\omega C)$ at high frequencies
                             $H(j\omega) = (j\omega RC - \omega^2 LC) / (j\omega RC + 1 - \omega^2 LC)$
                               $H(j\omega) \rightarrow -\omega^2 LC/-\omega^2 LC \rightarrow 1$ at $\omega \rightarrow \infty$
                               $\angle H(j\omega) \rightarrow \tan^{-1}(0/1)$ or $\angle$(+real) at $\omega \rightarrow \infty$
                               this is case(b)
                               $\angle H(\omega) = 0$ at $\omega \rightarrow \infty$

                               $H(j\omega) = j\omega L / (R + j\omega L + 1/j\omega C)$ at low frequencies
                               $H(j\omega) = (-\omega^2 LC) / (j\omega RC + 1 - \omega^2 LC)$
                               $H(j\omega) \rightarrow -\omega^2 LC$ at $\omega \rightarrow 0$
                               $\angle H(j\omega) \rightarrow \tan^{-1}(0/-\omega^2 LC)$ or $\angle$(- real) at $\omega \rightarrow 0$
                               this is case(d)
                               $\angle H(\omega) = \pi$ at $\omega \rightarrow 0$

                               $H(j\omega) = j\omega RC / (j\omega RC + 1)$ at the corner frequency

$\angle$ H(jω) = $\angle$(numerator) - $\angle$(denominator)

$\angle$ H(jω) = $\tan^{-1}$(jωRC/0) - $\tan^{-1}$(jωRC/1) = $\angle$(+j)-$\angle$(jωRC+1)

this is case(c) – case(f) iff ω_c = 1/RC

$\angle$ H(jω_c) = π/2 - $\tan^{-1}$(1/1) = π/2 - π/4 at ω_c = 1/RC

$\angle$ H(jω_c) = π/4 at ω_c = 1/RC

Appendix B – More on Taking Limits

You must be able to take limits in order to use transfer functions effectively. Basically, to take a limit as $\omega \to 0$ or $\omega \to \infty$, you must determine the dominant term in both the numerator and denominator and then consider the value of the ratio as the function approaches the limit. Note that when you first write out a transfer function (by looking at the circuit), it is often not in the best form for taking a limit. Multiplying all terms by (jωC) usually puts it in a form where there are no fractions in the numerator and denominator. More complex circuits may require more reduction. When you take the limit, try considering which of the two forms makes it easiest to understand. It might be either one. I like the one below.

When the transfer function has the general form:

$$\frac{(A\omega^2 + B\omega + C) + j(D\omega^2 + E\omega + F)}{(G\omega^2 + H\omega + I) + j(J\omega^2 + K\omega + L)}$$

To find the dominant term as $\omega \to 0$, look for the lowest power of ω in the numerator and the lowest power in the denominator.

   example:

$$\frac{R + j\omega L}{R + j\omega L + 1/j\omega C}$$ Next, multiply num and den by jωC.

$$\frac{j\omega RC - \omega^2 LC}{j\omega RC - \omega^2 LC + 1}$$ Next, find dominant terms as $\omega \to 0$.

$$\frac{j\omega RC}{1}$$ Reduce.

jωRC   Use this to approximate H(jω) at $\omega \to 0$.

To find the dominant term as $\omega \to \infty$, look for the highest power of ω in the denominator and the highest power in the numerator.

   example:

$$\frac{R}{R + j\omega L + 1/j\omega C}$$ Next, multiply num and den by jωC.

$$\frac{j\omega RC}{j\omega RC - \omega^2 LC + 1}$$ Next, find dominant terms as $\omega \rightarrow \infty$.

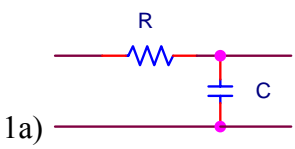$$\frac{j\omega RC}{\omega^2 LC}$$ Reduce.
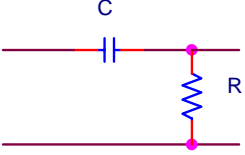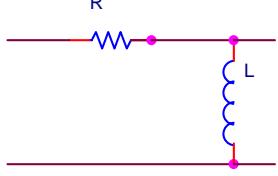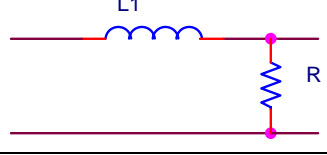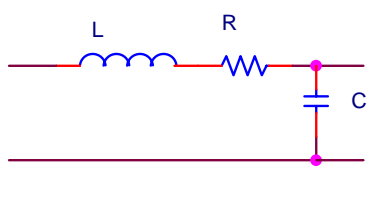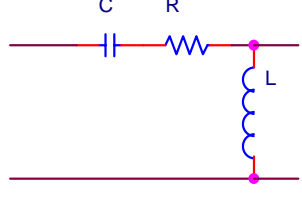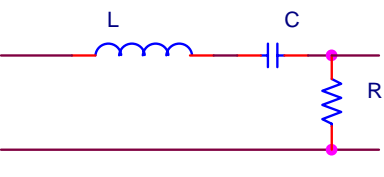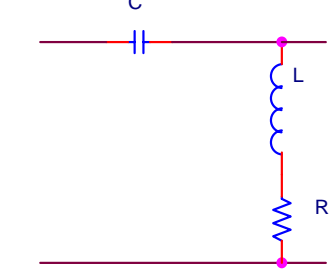
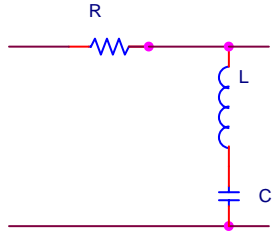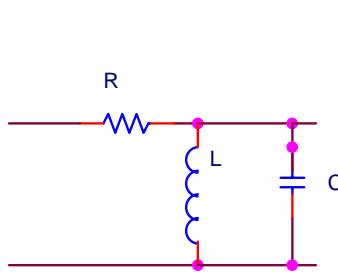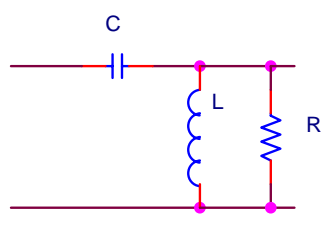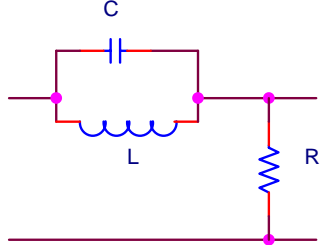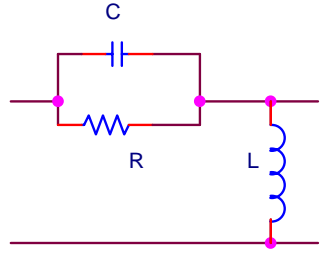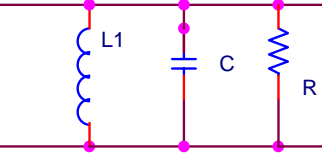$$\frac{jR}{\omega L}$$ This can be used for $H(j\omega)$ at $\omega \rightarrow \infty$.

Once you have the dominant term for both the numerator and the denominator, you can decide how the function behaves as ω approaches the desired limit. I made a chart of the different cases, as this is easier on the computer.

limit	dominant term in numerator	dominant term in denominator	limit approaches	comments
0	$1 / j\omega C$	$1 / j\omega C$	1 as $\omega \rightarrow 0$	H α 1
∞	$1 / j\omega C$	$1 / j\omega C$	1 as $\omega \rightarrow \infty$	H α 1
0	$1 / j\omega^2 C$	$1 / j\omega^2 C$	1 as $\omega^2 \rightarrow 0$	H α 1
∞	$1 / j\omega^2 C$	$1 / j\omega^2 C$	1 as $\omega^2 \rightarrow \infty$	H α 1
0	$j\omega RC$	1	0 as $\omega \rightarrow 0$	H α ω
∞	$j\omega RC$	1	∞ as $\omega \rightarrow \infty$	H α ω
0	$j\omega^2 RC$	1	0 as $\omega^2 \rightarrow 0$	H α ω^2
∞	$j\omega^2 RC$	1	∞ as $\omega^2 \rightarrow \infty$	H α ω^2
0	1	$j\omega RC$	∞ as $\omega \rightarrow 0$	H α $1/\omega$ (ω^{-1})
∞	1	$j\omega RC$	0 as $\omega \rightarrow \infty$	H α $1/\omega$ (ω^{-1})
0	1	$j\omega^2 RC$	∞ as $\omega^2 \rightarrow 0$	H α $1/\omega^2$ (ω^{-2})
∞	1	$j\omega^2 RC$	0 as $\omega^2 \rightarrow \infty$	H α $1/\omega^2$ (ω^{-2})
0	$j\omega^2 RC$	$j\omega LC$	0 as $\omega \rightarrow 0$	H α ω
∞	$j\omega^2 RC$	$j\omega LC$	∞ as $\omega \rightarrow \infty$	H α ω
0	$j\omega RC$	$j\omega^2 LC$	∞ as $\omega \rightarrow 0$	H α $1/\omega$ (ω^{-1})
∞	$j\omega RC$	$j\omega^2 LC$	0 as $\omega \rightarrow \infty$	H α $1/\omega$ (ω^{-1})

Appendix C – Examples of Transfer Functions

CIRCUIT	Transfer Function	Resonant or Corner Frequency
1a)	$H = \dfrac{1/j\omega C}{R + 1/j\omega C} = \dfrac{1}{j\omega RC + 1}$	$\omega_c = \dfrac{1}{RC}$ $f_c = \dfrac{1}{2\pi RC}$

1b)	$$H = \frac{R}{\frac{1}{j\omega C} + R} = \frac{j\omega RC}{1 + j\omega RC}$$	$$\omega_C = \frac{1}{RC}$$ $$f_C = \frac{1}{2\pi RC}$$
2a)	$$H = \frac{j\omega L}{R + j\omega L}$$	$$\omega_C = \frac{R}{L}$$ $$f_C = \frac{R}{2\pi L}$$
2b)	$$H = \frac{R}{R + j\omega L}$$	$$\omega_C = \frac{R}{L}$$ $$f_C = \frac{R}{2\pi L}$$
3a)	$$H = \frac{\frac{1}{j\omega C}}{j\omega L + R + \frac{1}{j\omega C}}$$ $$H = \frac{1}{-\omega^2 LC + j\omega RC + 1}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
3b)	$$H = \frac{j\omega L}{\frac{1}{j\omega C} + R + j\omega L}$$ $$H = \frac{-\omega^2 LC}{1 + j\omega RC - \omega^2 LC}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
3c)	$$H = \frac{R}{j\omega L + \frac{1}{j\omega C} + R}$$ $$H = \frac{j\omega RC}{-\omega^2 LC + 1 + j\omega RC}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
4a)	$$H = \frac{j\omega L + R}{\frac{1}{j\omega C} + j\omega L + R}$$ $$H = \frac{-\omega^2 LC + j\omega RC}{1 - \omega^2 LC + j\omega RC}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

4b) (circuit: R series, then L and C to ground)	$$H = \frac{j\omega L + \frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$ $$H = \frac{-\omega^2 LC + 1}{j\omega RC - \omega^2 LC + 1}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
5a) (circuit: R series, L parallel C)	$$H = \frac{\dfrac{j\omega L \times \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}}}{R + \dfrac{j\omega L \times \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}}}$$ $$H = \frac{j\omega L}{R - \omega^2 RLC + j\omega L}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
5b) (circuit: C series, L parallel R)	$$H = \frac{\dfrac{j\omega L \times R}{j\omega L + R}}{\frac{1}{j\omega C} + \dfrac{j\omega L \times R}{j\omega L + R}}$$ $$H = \frac{-\omega^2 RLC}{j\omega L + R - \omega^2 RLC}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
6a) (circuit: C parallel L series, R to ground)	$$H = \frac{R}{\dfrac{j\omega L \times \frac{1}{j\omega C}}{j\omega L + \frac{1}{j\omega C}} + R}$$ $$H = \frac{R - \omega^2 RLC}{j\omega L + R - \omega^2 RLC}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
6b) (circuit: C parallel R, L to ground)	$$H = \frac{j\omega L}{\dfrac{\frac{1}{j\omega C} \times R}{\frac{1}{j\omega C} + R} + j\omega L}$$ $$H == \frac{j\omega L - \omega^2 RLC}{R - \omega^2 RLC + j\omega L}$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$
7) (circuit: L1, C, R in parallel)	$$H = 1$$	$$\omega_0 = \frac{1}{\sqrt{LC}}$$ $$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Appendix D –Algebra Review

We want our transfer functions to be a ratio of two polynomials. In order to achieve this, we often have to eliminate fractions from the numerator and denominator of an expression.

Sometimes it is simply one fraction we need to get rid of. In this case, we multiply the top and the bottom by the denominator.

$$\frac{A + \frac{1}{B}}{C + \frac{D}{B}} = \frac{A + \frac{1}{B}}{C + \frac{D}{B}} \times \frac{B}{B} = \frac{AB + 1}{CB + D} \quad Note : \frac{B}{B} = 1$$

For example:

$$H = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \times \frac{j\omega C}{j\omega C} = \frac{1}{j\omega RC + 1}$$

Other times, we have more than one fraction in the denominator. This can be dealt with in one step by multiplying by the product of both denominators:

$$\frac{\frac{A}{C} + \frac{1}{B}}{E + \frac{D}{B}} = \frac{\frac{A}{C} + \frac{1}{B}}{E + \frac{D}{B}} \times \frac{BC}{BC} = \frac{AB + C}{EBC + DC} \quad Note : \frac{BC}{BC} = 1$$

or by eliminating one fraction at a time:

$$\frac{\frac{A}{C} + \frac{1}{B}}{E + \frac{D}{B}} = \frac{\frac{A}{C} + \frac{1}{B}}{E + \frac{D}{B}} \times \frac{B}{B} = \frac{\frac{AB}{C} + 1}{EB + D} = \frac{\frac{AB}{C} + 1}{EB + D} \times \frac{C}{C} = \frac{AB + C}{EBC + DC}$$

For example:

$$H = \frac{\frac{j\omega RL}{j\omega L + R}}{\frac{1}{j\omega C} + \frac{j\omega RL}{j\omega L + R}} = \frac{\frac{j\omega RL}{j\omega L + R}}{\frac{1}{j\omega C} + \frac{j\omega RL}{j\omega L + R}} \times \frac{j\omega L + R}{j\omega L + R} = \frac{j\omega RL}{\frac{j\omega L + R}{j\omega C} + j\omega RL}$$

$$= \frac{j\omega RL}{\frac{j\omega L + R}{j\omega C} + j\omega RL} \times \frac{j\omega C}{j\omega C} = \frac{-\omega^2 RLC}{j\omega L + R - \omega^2 RLC}$$