

Dynamic Systems Overview

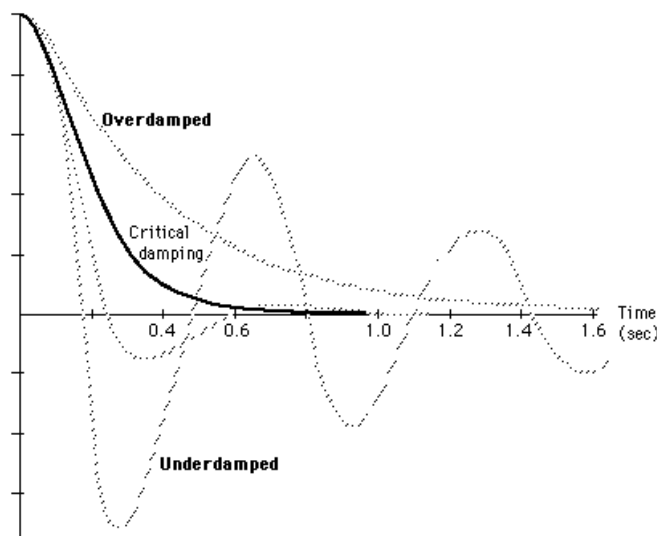
The [cantilever](#) beam and the [RLC circuit](#) are two examples of second order dynamic systems. (Note that all underlined terms can be found in Wikipedia.) Second order systems (so-called because their behavior is determined by a second-order differential equation) have two energy states that can trade system energy back and forth (e.g. potential and kinetic, capacitive and inductive, etc.) so that the parameters of the system will oscillate with time. There is usually also some mechanism for energy loss (e.g. friction, resistance, etc.) so that oscillations will damp out with time. If we attempt to change the state of a second order system (e.g. pluck the cantilever beam, change the voltage on a capacitor, etc.) we will see three types of behavior, depending on how much loss there is. Systems with loss are called damped systems because any disturbance will be damped out. The three types of behavior are three types of [damping](#).

Low Loss: If the loss is small (i.e. beam friction is small or RLC resistance is small), the system will oscillate freely for quite some time, but eventually the oscillation will end when all of the energy turns to heat through friction or resistance. This state is called ***under damped***. It is only under conditions where the system is under damped that an oscillation can be observed.

High Loss: If the loss is large, the system will not oscillate at all, but observable parameters will decay exponentially. This state is called ***over damped***. It can take quite a long time to reach equilibrium when the damping is large.

No Loss: If there is no loss at all, the system will oscillate forever. For circuits, this is the LC case with no resistance at all. There has to be an energy dissipation component to produce decay.

Critical Damping: If the loss is just right, the disturbance in the system will decay away in the shortest time. Critical damping also separates the operating regions where the system is over damped and under damped. The plot below shows some generic system parameter moving from one state to another under the three damping conditions.

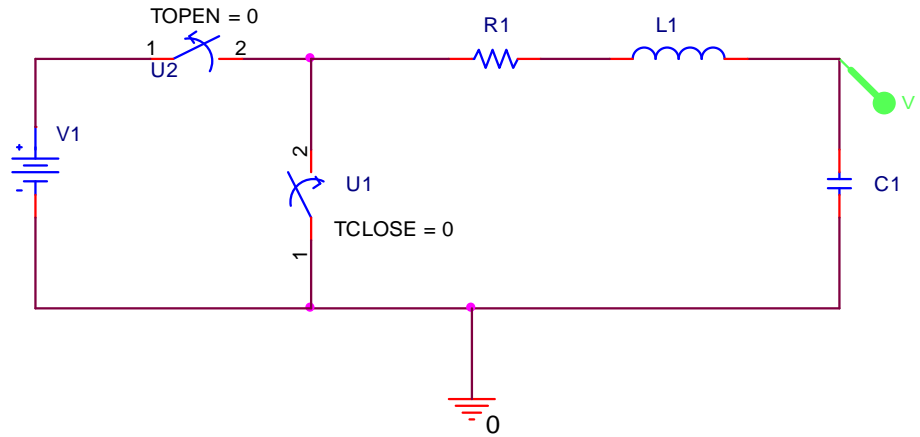


Damping Constant and Natural Frequency for Harmonic Oscillators

RLC Circuit

For the voltage across the capacitor V :

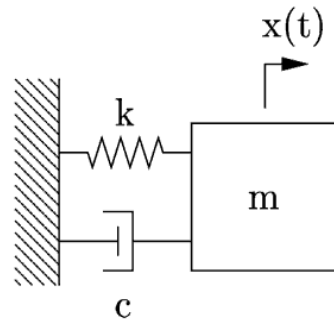
$$L\ddot{V} + R\dot{V} + \frac{1}{C}V = 0$$



Spring Mass System

For the linear displacement of the mass x ,

$$m\ddot{x} + c\dot{x} + kx = 0$$



General Form

$$\ddot{V} + 2\alpha\dot{V} + \omega_o^2 V = 0$$

This form works for both of the examples of harmonic oscillators. It has been left in the form of a voltage equation for convenience.

Solution

We know that the general form of the solution must be a decaying sinusoid, if for no other reason, because we have observed it in the classroom. Such an expression looks like

$V(t) = V_o e^{-\alpha t} \cos \omega t$. Note that we have assumed also at this point that the decay constant α is the same constant we find in the general form of the harmonic oscillator equation. We will prove that this is indeed the case by finding the frequency ω . Note that the frequency term in the general form is a constant ω_0 and not the actual frequency of oscillation ω . We will see that this is necessary.

- 1) $V(t) = V_o e^{-\alpha t} \cos \omega t$
- 2) $\dot{V}(t) = -\alpha V_o e^{-\alpha t} \cos \omega t - \omega V_o e^{-\alpha t} \sin \omega t$
- 3) $\ddot{V}(t) = \alpha^2 V_o e^{-\alpha t} \cos \omega t + \alpha \omega V_o e^{-\alpha t} \sin \omega t + \alpha \omega V_o e^{-\alpha t} \sin \omega t - \omega^2 V_o e^{-\alpha t} \cos \omega t$
or $\ddot{V}(t) = (\alpha^2 - \omega^2) V_o e^{-\alpha t} \cos \omega t + 2\alpha \omega V_o e^{-\alpha t} \sin \omega t$

Inserting these expressions into the general form of the harmonic oscillator.

$$(\alpha^2 - \omega^2)V_0 e^{-\alpha t} \cos \omega t - 2\alpha\omega V_0 e^{-\alpha t} \sin \omega t + 2\alpha(-\alpha V_0 e^{-\alpha t} \cos \omega t - \omega V_0 e^{-\alpha t} \sin \omega t) + \omega_0^2 V_0 e^{-\alpha t} \cos \omega t = 0$$

$$(\alpha^2 - \omega^2)V_0 e^{-\alpha t} \cos \omega t + 2\alpha(-\alpha V_0 e^{-\alpha t} \cos \omega t) + \omega_0^2 V_0 e^{-\alpha t} \cos \omega t = 0$$

$$(\alpha^2 - \omega^2) + 2\alpha(-\alpha) + \omega_0^2 = 0$$

$$-\omega^2 - \alpha^2 + \omega_0^2 = 0$$

$$\omega^2 = -\alpha^2 + \omega_0^2$$

$$\omega = \sqrt{\omega_0^2 - \alpha^2}$$

Thus, the α term is indeed the damping constant. The frequency of oscillation depends on the damping. The greater the damping, the lower the frequency. This is a general characteristic of harmonic oscillators. The more the damping, the more the loss per cycle and the slower the oscillation. The conclusion is that we have the correct form for the decaying sinusoid as long as the frequency is given by the last expression. For the highly under damped cases we are considering in this course, the damping term is small so $\omega \approx \omega_0$, which is what we have been using to find the frequency.

Note that a decaying sinusoid is the shape we expect for an under damped system (under damped means that the decay per cycle is small). The frequency formula show that the system is under damped when the damping constant squared is smaller than the natural frequency squared.

$\omega = \sqrt{\omega_0^2 - \alpha^2}$ If the damping constant equals the natural frequency, then the damping is critical. If the damping constant exceeds the natural frequency, then the system is over damped.

