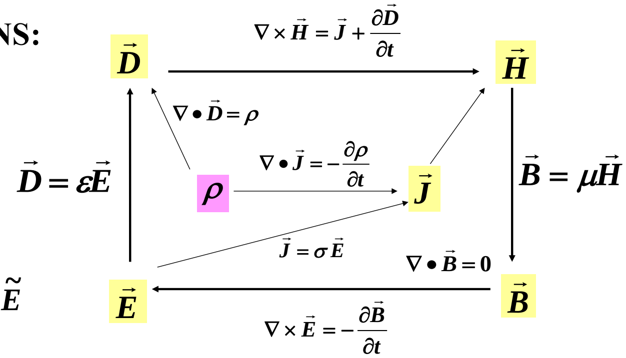
# Electromagnetic (EM) Waves

**TIME HARMONIC FORM OF MAXWELL'S EQUATIONS:**




$$\nabla\bullet\vec{E}=\rho_V/\varepsilon \qquad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \nabla\bullet\vec{H}=0 \qquad \nabla\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t}$$

$$\begin{cases}\vec{D}=\varepsilon\vec{E}\\ \vec{B}=\mu\vec{H}\\ \vec{J}=\sigma\vec{E}\end{cases} \Rightarrow$$

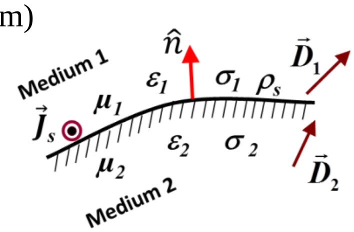
$$\begin{cases}\nabla\bullet\tilde{E}=\tilde{\rho}_v/\varepsilon\\ \nabla\times\tilde{E}=-j\omega\mu\tilde{H}\\ \nabla\bullet\tilde{H}=0\\ \nabla\times\tilde{H}=j\omega(\varepsilon-j\frac{\sigma}{\omega})\tilde{E}=j\omega\varepsilon_c\tilde{E}\end{cases}$$

$$\vec{E}(x,y,z,t)=\Re e\{\tilde{E}(x,y,z)e^{j\omega t}\} \qquad \varepsilon_c=\varepsilon-j\frac{\sigma}{\omega}=\varepsilon'-j\varepsilon''$$

**CONSTANTS:** $\mu_o\approx 4\pi\times10^{-7}$ **H/m** $\quad \varepsilon_o\approx\frac{1}{36\pi}\times10^{-9}$ **F/m** $\quad c=3\times10^{8}$ **m/s**

**BOUNDARY CONDITIONS** (Hold for both general time variation and phasor form)

| | |
|---|---|
| $E_{1t}=E_{2t}$ or $\hat{n}\times(\vec{E}_1-\vec{E}_2)=0$ | $-H_{1t}+H_{2t}=J_s$ or $\hat{n}\times(\vec{H}_1-\vec{H}_2)=\vec{J}_s$ |
| $D_{1n}-D_{2n}=\rho_s$ or $\hat{n}\bullet(\vec{D}_1-\vec{D}_2)=\rho_s$ | $B_{1n}=B_{2n}$ or $\hat{n}\bullet(\vec{B}_1-\vec{B}_2)=0$ |




**PLANE WAVE IN SOURCE-FREE REGION** (traveling in z-direction)

$$\tilde{E}(z)=\hat{x}\tilde{E}_x(z)+\hat{y}\tilde{E}_y(z) \qquad \tilde{E}_x(z)=\tilde{E}_x^+(z)+\tilde{E}_x^-(z)=E_{xo}^+e^{-(\alpha+j\beta)z}+E_{xo}^-e^{(\alpha+j\beta)z}$$

$$\tilde{H}(z)=\hat{x}\tilde{H}_x(z)+\hat{y}\tilde{H}_y(z) \qquad \gamma=\alpha+j\beta \qquad \gamma^2=-\omega^2\mu\varepsilon_c=-\omega^2\mu(\varepsilon'-j\varepsilon'')$$

**Speed:** $u_p=\frac{\omega}{\beta}=f\lambda=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{c}{\sqrt{\varepsilon_r}}$; **Wavelength:** $\lambda=\frac{2\pi}{\beta}=\frac{2\pi}{k}=\frac{\lambda_o}{\sqrt{\varepsilon_r}}=\frac{\lambda_o}{n}$; **Skin Depth:** $\delta_s=\frac{1}{\alpha}$

$$\tilde{E}=-\eta_c\hat{z}\times\tilde{H} \qquad \tilde{H}=\frac{1}{\eta_c}\hat{z}\times\tilde{E} \qquad \eta_c=\sqrt{\frac{\mu}{\varepsilon_c}}=\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \qquad \eta_o=\sqrt{\frac{\mu_o}{\varepsilon_o}}=120\pi\ \Omega$$

**MATERIAL PROPERTIES:**

| | Any Medium | Lossless Medium ($\sigma=0$) | Low-loss Medium ($\varepsilon''/\varepsilon'\ll1$) | Good Conductor ($\varepsilon''/\varepsilon'\gg1$) | Units |
|---|---|---|---|---|---|
| $\alpha=$ | $\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}-1\right]\right]^{1/2}$ | $0$ | $\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$ | $\sqrt{\pi f\mu\sigma}$ | (Np/m) |
| $\beta=$ | $\omega\left[\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2}+1\right]\right]^{1/2}$ | $\omega\sqrt{\mu\varepsilon}$ | $\omega\sqrt{\mu\varepsilon}$ | $\sqrt{\pi f\mu\sigma}$ | (rad/m) |
| $\eta_c=$ | $\sqrt{\frac{\mu}{\varepsilon'}}\left(1-j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2}$ | $\sqrt{\frac{\mu}{\varepsilon}}$ | $\sqrt{\frac{\mu}{\varepsilon}}$ | $(1+j)\frac{\alpha}{\sigma}$ | ($\Omega$) |
| $u_p=$ | $\omega/\beta$ | $1/\sqrt{\mu\varepsilon}$ | $1/\sqrt{\mu\varepsilon}$ | $\sqrt{4\pi f/\mu\sigma}$ | (m/s) |
| $\lambda=$ | $2\pi/\beta=u_p/f$ | $u_p/f$ | $u_p/f$ | $u_p/f$ | (m) |

Notes: $\varepsilon'=\varepsilon$; $\varepsilon''=\sigma/\omega$; in free space, $\varepsilon=\varepsilon_0$, $\mu=\mu_0$; in practice, a low-loss medium if $\varepsilon''/\varepsilon'=\sigma/\omega\varepsilon<0.01$ and a good conducting medium if $\varepsilon''/\varepsilon'>100$.

**Poynting vector**: $\vec{S} = \vec{E} \times \vec{H}$ [W/m²];

Average power density: $\vec{S}_{av} = \frac{1}{2} Re\{\tilde{E} \times \tilde{H}^*\} = \hat{z} \frac{|\tilde{E}|^2}{2|\eta_c|} \cos\theta_{\eta_c}$ [W/m²]

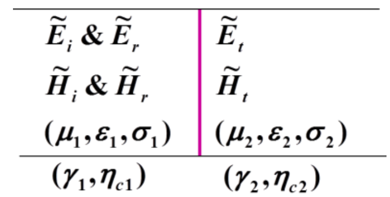
Lossless media: Wavenumber $k = \beta = \omega\sqrt{\mu\varepsilon}$

**POLARIZATION**: $\tilde{E}(z) = (\hat{x}a_x + \hat{y}a_y e^{j\delta})e^{-j\kappa z}$; $\vec{E}(z,t) = \hat{x}a_x \cos(\omega t - \kappa z) + \hat{y}a_y \cos(\omega t - \kappa z + \delta)$

***Linear*: δ=0 or π; *Circular*: $a_x = a_y$ & δ = ±π/2 (LHC/RHC); *Elliptical*:** General

**NORMAL INCIDENCE:**

$$\begin{cases} \Gamma = \dfrac{E_o^r}{E_o^i} = \dfrac{\eta_{c2} - \eta_{c1}}{\eta_{c2} + \eta_{c1}} \\ \tau = \dfrac{E_o^t}{E_o^i} = \dfrac{2\eta_{c2}}{\eta_{c2} + \eta_{c1}} = 1 + \Gamma \end{cases} \qquad \begin{cases} \vec{S}_{av1}(z) = \hat{z}\dfrac{|E_o^i|^2}{2|\eta_c|}\left[\left(e^{-2\alpha z} - |\Gamma|^2 e^{2\alpha z}\right)\cos\theta_{\eta_c} - 2|\Gamma|\sin(2\beta z + \theta_\Gamma)\sin\theta_{\eta_c}\right] \\ \vec{S}_{av2}(z) = \hat{z}\dfrac{|E_o^i|^2}{2|\eta_{c2}|}|\tau|^2 e^{-2\alpha_2 z}\cos\theta_{\eta_{c2}} \end{cases}$$




For *x*-polarized EM wave traveling in z-direction:

$$\begin{cases} \tilde{E}_1(z) = \tilde{E}^i + \tilde{E}^r = \hat{x}E_o^i(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}) \\ \tilde{H}_1(z) = \tilde{H}^i + \tilde{H}^r = \hat{y}\dfrac{E_o^i}{\eta_{c1}}(e^{-\gamma_1 z} - \Gamma e^{\gamma_1 z}) \end{cases} \qquad \begin{cases} \tilde{E}_t(z) = \hat{x}\tau E_o^i e^{-\gamma_2 z} \\ \tilde{H}_t(z) = \hat{y}\tau\dfrac{E_o^i}{\eta_{c2}} e^{-\gamma_2 z} \end{cases}$$

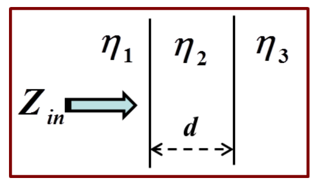
Multiple Boundaries:

Wave impedance

$$Z(z) = \frac{\tilde{E}(z)}{\tilde{H}(z)} = \eta\frac{1+\Gamma(z)}{1-\Gamma(z)}$$

Input impedance

$$Z_{in} = \eta_{in} = \eta_2 \frac{\eta_3 + j\eta_2 \tan\beta_2 d}{\eta_2 + j\eta_3 \tan\beta_2 d}$$




**OBLIQUE INCIDENCE:**

Arbitrary ($\vec{k}$) Direction: $\tilde{E} = \vec{E}_m e^{-j\vec{k}\cdot\vec{r}} = \hat{a}_E E_m e^{-jkx_i}$ $\quad \tilde{H} = \frac{1}{\eta}\hat{k} \times \tilde{E} = \hat{a}_H \frac{E_m}{\eta} e^{-jkx_i}$

*e.g., for the figure below,* $\tilde{E}_\perp^i = \hat{y}E_{\perp o}^i e^{-jk_i x_i}$, $\tilde{H}_\perp^i = \hat{y}_i \frac{E_{\perp o}^i}{\eta_i} e^{-jk_i x_i}$,

where $x_i = \hat{k}_i \bullet \vec{r} = x\sin\theta_i + z\cos\theta_i$, $\hat{y}_i = -\hat{x}\cos\theta_i + \hat{z}\sin\theta_i$.

**Snell's Law**: $\theta_i = \theta_r \equiv \theta_1$ $\qquad$ Index of Refraction: $n = \sqrt{\varepsilon_r}$ (lossless non-magnetic)

$$k_1 \sin\theta_i = k_2 \sin\theta_t \quad \text{or} \quad n_1 \sin\theta_i = n_2 \sin\theta_t$$

**Perpendicular Polarization:**

$$\Gamma_\perp = \frac{E_{\perp 0}^r}{E_{\perp 0}^i} = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i + \eta_1\cos\theta_t}$$

$$\tau_\perp = \frac{E_{\perp 0}^t}{E_{\perp 0}^i} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i + \eta_1\cos\theta_t} = 1 + \Gamma_\perp$$

**Parallel Polarization:**

$$\Gamma_{||} = \frac{E_{||0}^r}{E_{||0}^i} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\tau_{||} = \frac{E_{||0}^t}{E_{||0}^i} = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = (1 + \Gamma_{||})\frac{\cos\theta_i}{\cos\theta_t}$$

**Power Reflectivity and Transmissivity**:

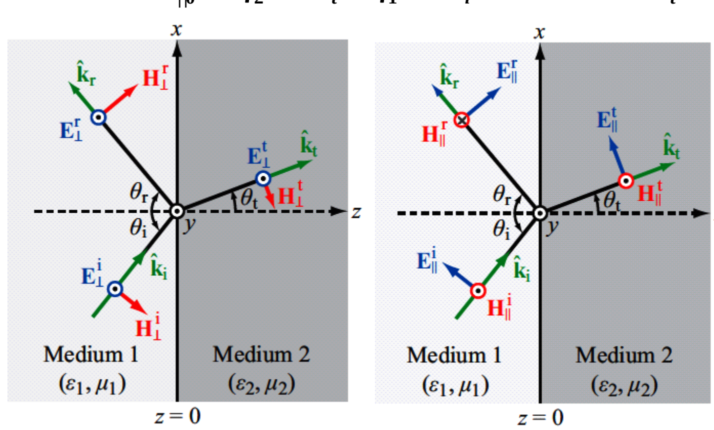
**R= $|\Gamma|^2$ $\quad$ T=1-R**

**Critical Angle**: $\sin\theta_c = \dfrac{n_2}{n_1} = \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}$

**Brewster Angle**: $\theta_{B||} = \sin^{-1}\dfrac{1}{\sqrt{1 + \varepsilon_1/\varepsilon_2}}$

**(for $\mu_1 = \mu_2$)**

$$= \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$




(a) Perpendicular polarization $\qquad$ (b) Parallel polarization