10

s-Plane analysis: Root locus

By now the reader hopefully has been convinced of the advantages of knowing the roots of a feedback system's characteristic equation in terms of its open-loop parameters (poles, zeros, gains, etc.). Accordingly, we shall proceed to establish analytic methods for finding these roots using either the complex-frequency domain (the s-plane) or the real-frequency domain, as covered in this and the next chapter, respectively.

The principal vehicle used in the s-plane is the root locus, which yields the exact locations of all of the system's roots. A less involved method, giving only a yes-or-no answer to the question of a system's stability, is based upon Routh's criterion. Both are considered here. To demonstrate the utility of these tools and to give the reader a little practice before turning him loose, we shall conclude by revisiting the satellite-attitude control problem introduced in the previous chapter.

10.1 THE ROOT-LOCUS METHOD

In a nutshell, the root-locus is a plot in the s-plane of all possible locations that the roots of a closed-loop system's characteristic equation can have as a specific parameter is varied, usually from zero to infinity. Because of the central role played by the characteristic equation we shall start by giving a variety of different forms that it can take, all of which will be useful in understanding the root locus and in deriving rules for rapidly plotting it.

The characteristic equation

Recall from Section 9.2 that the basic single-loop feedback system represented by the block diagram in Figure 10.1 has the characteristic

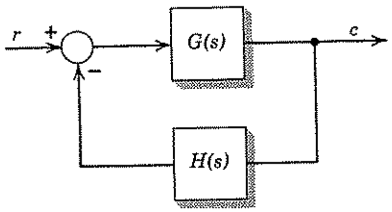

Figure 10.1.

equation

$$D_G(s)D_H(s) + N_G(s)N_H(s) = 0 \tag{1}$$

where $N_G(s)/D_G(s) \triangleq G(s) \triangleq G_c(s)G_p(s)$ and $N_H(s)/D_H(s) \triangleq H(s)$. An alternate form of (1) can be obtained by dividing it by the polynomial $D_G(s)D_H(s)$, resulting in

$$1 + \frac{N_G(s)N_H(s)}{D_G(s)D_H(s)} = 0 \tag{2}$$

which can be written more compactly as

$$1 + G(s)H(s) = 0 \tag{3}$$

Although (3) could perhaps have been obtained by inspection, the reader should note that its left-hand side is not a polynomial; however, the values of s for which it is satisfied will be the same as those for which (1) is satisfied provided that the numerator and denominator polynomials in (2) have no common factors.

In anticipation of the equations we shall need in constructing the root locus, we rewrite the rational function $G(s)H(s)$ as a ratio of *factored polynomials* such that

$$G(s)H(s) = K \underbrace{\frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{k=1}^{n} (s - p_k)}}_{F(s)} \tag{4}$$

in which case the characteristic equation can be written as

$$1 + KF(s) = 0 \tag{5}$$

where

$$F(s) \triangleq \frac{\prod_{i=1}^{m} (s - z_i)}{\prod_{k=1}^{n} (s - p_k)} \tag{6}$$

The parameter K is the constant necessary for the coefficients of the highest powers of s in both numerator and denominator of $F(s)$ to be unity and it is necessarily real. It is referred to as the *root-locus gain* and will play an important role in our development and subsequent use of the root locus. From (6), it is apparent that $F(s)$ has the combined poles and zeros of the individual transfer functions of all of the dynamic elements contained within the closed-loop. Furthermore, because of the rational-function form of $F(s)$ we can write

$$|F(s)| = \frac{\prod_{i=1}^{m} |s - z_i|}{\prod_{k=1}^{n} |s - p_k|} \tag{7}$$

and

$$\arg[F(s)] = \sum_{i=1}^{m} \arg[s - z_i] - \sum_{k=1}^{n} \arg[s - p_k] \tag{8}$$

where the terms $(s - z_i)$ may be represented by vectors from the zeros z_i to the point s at which $F(s)$ is being evaluated and likewise for the poles. A simple example corresponding to $n = m = 2$ is shown in Figure 10.2. For instance, both the magnitude and argument of $(s_0 - z_1)$ can be measured on the figure, giving $|s_0 - z_1| = 2$ and $\arg[s_0 - z_1] = 90°$. Carrying out the process for the other zero and the two poles and substituting into (7) and (8)

$$|F(s_0)| = \frac{2.00 \times 3.20}{2.25 \times 3.65} = 0.778$$

$$\arg[F(s_0)] = (90 + 38) - (26 + 56) = 46°$$

The other forms of the characteristic equation that we shall need can be derived from (5) and (6) and are

$$\prod_{k=1}^{n} (s - p_k) + K \prod_{i=1}^{m} (s - z_i) = 0 \tag{9}$$

$$F(s) = -\frac{1}{K} \tag{10}$$

$$\frac{1}{K} \prod_{k=1}^{n} (s - p_k) + \prod_{i=1}^{m} (s - z_i) = 0 \tag{11}$$

the last being contingent upon $K \neq 0$.

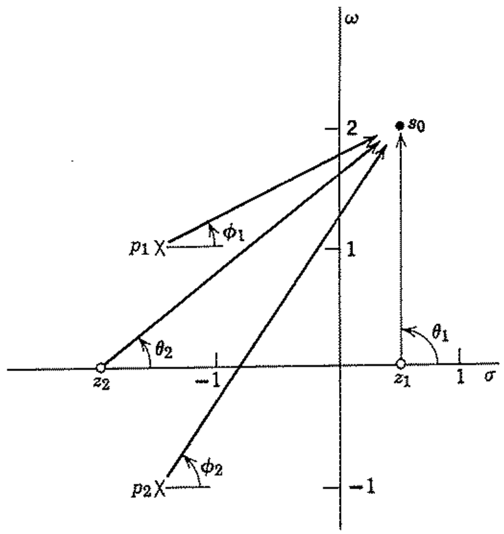

Figure 10.2. Evaluating $F(s)=\dfrac{(s-0.5)(s+2)}{(s+1.5-j)(s+1.5+j)}$ at $s=0.5+j2$.

Angle and magnitude criteria

We assume that we know the open-loop transfer function $G(s)H(s)$ – this will be needed in *factored* form – and that the root-locus gain K can be thought of as potentially variable from 0 to ∞ (sometimes from $-\infty$ to 0 or from $-\infty$ to $+\infty$). Because (10) is a valid form of the characteristic equation in that its roots are the roots of (1), the problem of finding the roots of the characteristic equation (closed-loop poles) is equivalent to finding the solutions to the algebraic equation

$$F(s)=-\frac{1}{K}$$

where K is real. Expressing both $F(s)$ and K in polar form gives yet another version of the characteristic equation, namely

$$\underbrace{|F(s)|e^{j\arg[F(s)]}}_{F(s)}=\underbrace{\frac{1}{|K|}e^{jq180^\circ}}_{-1/K} \tag{12}$$

where q is an *odd integer* if $K>0$ and an *even integer* if $K<0$.

In the light of (12), any value of s that is to be a closed-loop pole must simultaneously satisfy the *angle criterion*

$$\arg[F(s)]=q\,180^\circ \tag{13}$$

and the *magnitude criterion*

$$|F(s)|=\frac{1}{|K|} \tag{14}$$

The existence of these two requirements suggests that the roots of the system's closed-loop characteristic equation corresponding to specific parameter values may be obtained by the following sequence of operations:

1. Find *all* values of s that satisfy the angle criterion, (13), the plot in the s-plane of these values being the *root locus.*

2. Find the specific values of s on the root locus that satisfy the magnitude criterion, (14), or, as an alternative, find the value of K that will cause a specific point on the locus to be a root.

Although this two-stage process seems perhaps like a roundabout way of solving the problem, it turns out to have several important advantages over other methods. For one thing, by breaking it into the two parts the task of finding the roots of the characteristic equation is greatly simplified, as we shall see shortly. For another, the method has the attractive feature that by applying the angle criterion first we are finding *all* of the possible root locations for any $K\neq 0$ (the case $K=0$ is easily included, although the magnitude criterion is undefined there). Because the root-locus gain K usually depends on a physical parameter that is readily adjusted, e.g., an amplifier gain, we can make effective use of the root locus as a design tool to suggest those values of gain that yield preferred closed-loop roots. Finally, if we cannot find a value of gain that yields satisfactory roots, e.g., stable and well damped, an examination of the root-locus plot will usually provide insight as to how other system parameters should be varied or how the structure of the system should be altered, such as by adding filters or feeding back different signals.

Example 10.1

To demonstrate the above notions with a very elementary example, we shall draw the root locus for the system shown in Figure 10.3*a* for which $F(s)=1/s$ and locate the closed-loop root(s) for the specific value

$K = 10$. Substituting for $F(s)$ in (13) and (14), the angle and magnitude criteria become arg $[F(s)] = -\text{arg}\ [s]$ and $|F(s)| = 1/|s|$, respectively.

Considering the angle criterion with $K > 0$, q is an odd integer. Consequently those points that lie on the root locus are those values of s for which arg $[F(s)]$ is an odd multiple of 180°, e.g., ±180°, ±540°. However, this cannot happen unless arg $[s]$ is also an odd multiple of 180°, which is to say that s is restricted to points on the negative-real axis. Hence, the root locus for the system with $K > 0$ must be as shown in Figure 10.3*b*.

Having obtained the root locus for $K > 0$, we can identify the specific closed-loop root corresponding to $K = 10$ by applying the magnitude criterion, obtaining $|s| = 10$. Since $s = -10$, denoted by the triangle in Figure 10.3*c*, is the only point in the entire s-plane that satisfies both the angle and the magnitude criteria, it must be the single root of the closed-loop system. A glance back to Example 9.2 will substantiate the validity of the above conclusions.

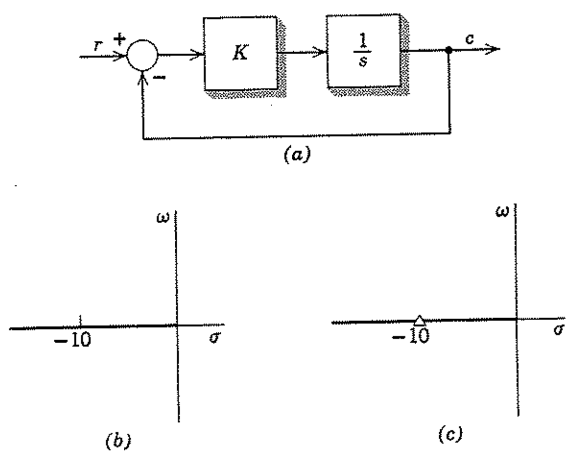

Figure 10.3. First-order system. (*a*) Block diagram. (*b*) 180° locus. (*c*) Root location, $K = 10$.

Construction of the 180° root locus

The procedure followed in the previous example of solving analytically for $|F(s)|$ and arg $[F(s)]$ is not at all practical for other than trivial problems and will not be pursued further. Rather, a number of rules exist that allow one to *sketch* certain salient features of the root locus with only a modest effort, even for rather complex systems. Also, a device called a "Spirule"† has been developed which, with a bit of practice, can be used to obtain a more accurate *plot* of the root locus, should it be required. Finally, digital-computer programs‡ have been written for achieving high accuracy and relieving the engineer of the plotting burden.

We shall present the rules for constructing a number of the important features of the root locus, deferring their derivations until the following section. Because the angle criterion requires that arg $[F(s)]$ be an *odd* multiple of 180° when $K > 0$, we refer to the locus corresponding to $K \geq 0$ as the *180° locus*. When $K < 0$ the angle criterion must yield an *even* multiple of 180°, e.g., 0°, ±360°; hence the designation *0° locus* is used for $K \leq 0$. In the interest of simplicity, we shall work with the 180° locus for now and show the relatively minor modifications necessary for the 0° locus later.

It will be assumed that the n poles and m zeros of $F(s)$ are distinct, that no poles coincide with zeros, and that $m \leq n$ (the case of repeated poles and/or zeros will be taken up in Section 10.3). The basic rules for the construction of a 180° root locus ($K \geq 0$) are given below.

1. The locus has exactly n branches, where a branch is the path formed by any one root as K is varied continuously from zero to infinity.

2. The locus is symmetric with respect to the real axis of the s-plane.

3. Any point on the real axis is on the root locus if the total number of real poles and zeros to the right of that point is odd.

4. As K increases from 0, the n branches of the root locus depart from the poles of $F(s)$, one branch per pole.

5. As $K \to \infty$, m of the branches of the locus approach the finite zeros of $F(s)$, one branch per zero.

6. If $m < n$, there are $n-m$ branches of the root locus that approach infinity as $K \to \infty$; furthermore, they approach infinity asymptotic to the $n-m$ straight lines that intersect the real axis at the point

$$\sigma_0 = \frac{\sum_{k=1}^{n} p_k - \sum_{i=1}^{m} z_i}{n-m} \tag{15}$$

†Available from your bookstore or directly from The Spirule Co., 9728 El Venado, Whittier, California. See D'Azzo and Houpis (1966, App. D.) for a description of its use.

‡See Ash and Ash (1968).

and form the angles ψ_ν with the real axis where

$$\psi_\nu = \frac{180° + \nu 360°}{n-m} \qquad \nu = 0, 1, \ldots, (n-m-1) \tag{16}$$

7. The angle with which the locus departs from a complex open-loop pole p_J is given by

$$\phi_J = \sum_{i=1}^{m} \arg [p_J - z_i] - \sum_{\substack{k=1 \\ k \neq J}}^{n} \arg [p_J - p_k] + q180° \tag{17}$$

where q is an odd integer.

8. The angle at which the locus arrives at a complex open-loop zero z_J is given by

$$\theta_J = -\sum_{\substack{i=1 \\ i \neq J}}^{m} \arg [z_J - z_i] + \sum_{k=1}^{n} \arg [z_J - p_k] + q180° \tag{18}$$

where q is an odd integer.

Having applied the above rules to a given problem, enough of the root locus is usually known so that the remainder can be sketched to within a reasonable approximation. As a further refinement, one can always select a test point $\hat{s}$ that looks as if it should be close to the locus and evaluate the argument of $F(\hat{s})$. If $\arg [F(\hat{s})] = \pm 180°$, then $\hat{s}$ is on the locus. If not, a nearby point is selected and the argument reevaluated until the result is sufficiently close to $\pm 180°$ to consider the test point as being on the locus. It is in this phase of the graphical process that the Spirule is of greatest assistance, allowing for the rapid measurement and addition or subtraction of the angles of vectors to the test point $\hat{s}$ from the poles and zeros of $F(s)$.

Example 10.2

As an example that is only slightly less trivial than Example 10.1, we shall find the root locus corresponding to

$$F(s) = \frac{1}{s(s+2)}$$

where $0 \leq K \leq \infty$. As $F(s)$ has two poles (at $s = 0$ and -2) and no zeros, $n = 2$ and $m = 0$. Applying rules *5* and *6*, the locus has two branches, one emanating from $s = 0$ and the other from $s = -2$ for $K \approx 0$ and both going to infinity as $K \to \infty$. Applying rule *6*, the two branches approach infinity at angles of $\pm 90°$ with respect to the real axis, asymptotic to a vertical line that intersects the real axis at

$$\sigma_0 = \frac{p_1 + p_2}{n-m} = \frac{0-2}{2-0} = -1$$

Only that portion of the real axis between 0 and -2 is on the locus because *1*. there are no real poles or zeros of $F(s)$ to the right of $s = 0$; *2*. there is a single real pole to the right of $s = -2$; and *3*. there are two real poles to the right of all points to the left of $s = -2$ (rule *3*).

Drawing what we know at this point about the locus leads to the heavy lines shown in Figure 10.4*a*, which accounts for high and low values of K; however, the behavior of the locus for intermediate values of K during the transition from the real axis to the vertical asymptotes is as yet unknown. If we select any test point $\hat{s}$ on the vertical asymptote (Figure 10.4*a*) we see that $\phi_1 + \phi_2 = 180°$, which means that $\arg [F(\hat{s})] = -180°$ and that the test point must lie on the locus. Since the angle criterion is satisfied for any point on the vertical asymptote, the complete root locus must appear as in part (*b*).

Having found the root locus, we can apply the magnitude criterion if we wish to know the root locations for a specific nonnegative value of K. Alternatively, suppose we desire that the point $s_1 = -1 + j\sqrt{3}$ shown in part (*c*) be a root of the characteristic equation. We can use the magnitude

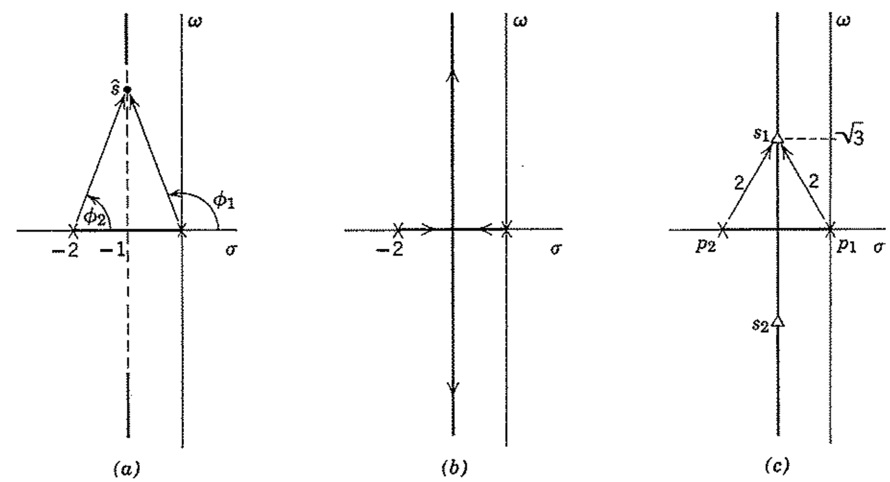

Figure 10.4. (*a*) Applying angle criterion at $\hat{s}$. (*b*) Complete locus. (*c*) Applying magnitude criterion at s_1.

criterion to solve for the required value of K. Constructing the vectors from p_1 and p_2 to s_1, both of which are of length two, it follows that

$$|F(s_1)| = \frac{1}{|s_1 - p_1| \times |s_1 - p_2|} = \frac{1}{2 \times 2} = \frac{1}{4}$$

which, from (14), indicates that K must be equal to 4 if s_1 is to be a closed-loop root. Because of the symmetry condition about the real axis, $s_2 = 1 - j\sqrt{3}$ will be the other root corresponding to $K = 4$.

For this simple example we can verify the result by directly computing the roots of $1 + KF(s) = 0$ with $K = 4$. Substituting for $F(s)$ gives $1 + [4/s(s+2)] = 0$ which, in polynomial form, becomes $s^2 + 2s + 4 = 0$ and has the required roots.

Example 10.3

The feedback system shown in Figure 10.5a represents a control system for which the process is unstable due to the open-loop pole at $s = 1$. By plotting the root locus we can find the range of the gain α such that all three closed-loop poles lie in the LHP. From the block diagram

$$G(s)H(s) = \frac{5\alpha}{(s-1)(s^2+2s+5)} = \underbrace{5\alpha}_{K}\Bigg[\underbrace{\frac{1}{(s-1)(s+1-j2)(s+1+j2)}}_{F(s)}\Bigg]$$

Applying the basic root-locus rules with $n = 3$ and $m = 0$, there are three branches emanating from the three poles of $F(s)$ for $K \approx 0$ and going to infinity as $K \to \infty$ ($\alpha \to \infty$). The real-axis segment to the left of $s = 1$ is on the root locus and the large-gain asymptotes intersect at

$$\sigma_0 = \frac{1 + (-1 + j2) + (-1 - j2)}{3} = -\frac{1}{3}$$

Using (16) with $\nu = 0$, 1, and 2 we see that the large-gain asymptotes make angles of 60°, 180°, and 300° with the real axis.

To calculate the angle of departure as the locus leaves the complex pole at $p_2 = -1 + j2$ we use (17) with $q = 1$ (actually, any odd integer will do) and $J = 2$. Thus,

$$\phi_2 = -\{\underbrace{\arg[p_2 - p_1]}_{135^\circ} + \underbrace{\arg[p_2 - p_3]}_{90^\circ}\} + 180^\circ = -45^\circ$$

Because of the symmetry property of the root locus, the angle of departure from p_3 is $\phi_3 = -\phi_2 = +45^\circ$.

Figure 10.5b shows the complete 180° root locus of the system, where the segments of the two branches leaving the complex poles have been determined for intermediate values of α by checking the angle criterion at several test points. For instance, $s_1 = j\sqrt{3}$ turns out to be the point at which the upper branch crosses into the RHP and application of the magnitude criterion gives

$$K_1 = \frac{1}{|F(s)|_{s=s_1}} = 2 \times 1.05 \times 3.80 = 8.00$$

where the lengths of the three vectors can be obtained graphically from the root locus. Relating the "cross-over" value of the root-locus gain K to the adjustable loop gain α, $\alpha_1 = 8.00/5 = 1.80$ since $K = 5\alpha$.

For very low values of the gain there will be a root on the positive real axis between the origin and $s = 1$. To find the value of K, denoted by K_2, at which the real branch of the locus crosses into the LHP we apply

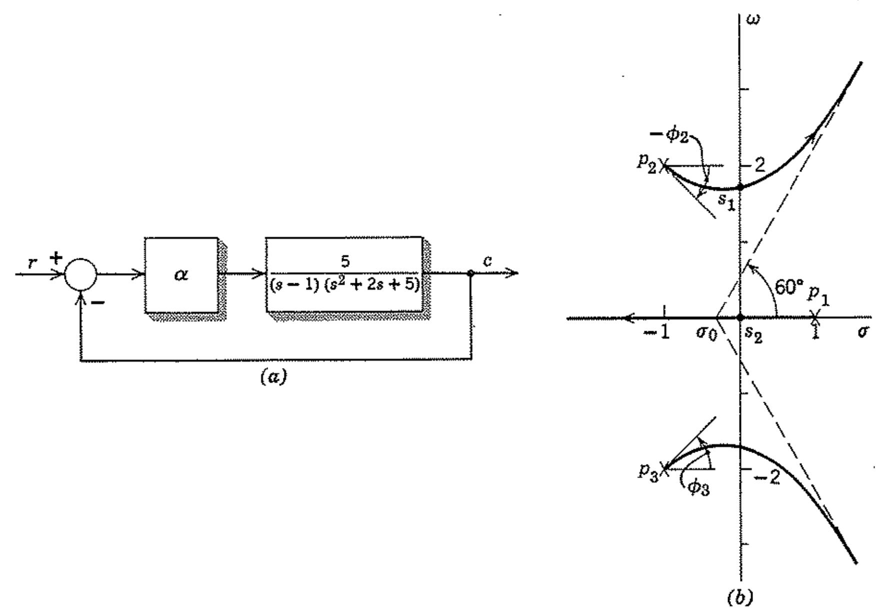

Figure 10.5. A third-order system without zeroes. (a) Block diagram. (b) Root locus.

the magnitude criterion at the point $s_2 = 0$, obtaining

$$K_2 = \frac{1}{|F(0)|} = 1 \times \sqrt{5} \times \sqrt{5} = 5$$

whence $\alpha_2 = 1.00$.

Combining the two cross-over values of α and using the root-locus plot as the basis for inferring the range of α that will result in all three roots being in the LHP, we see that the closed-loop system will be stable for $\alpha_2 < \alpha < \alpha_1$ or, using numerical values, for $1.00 < \alpha < 1.80$.

Example 10.4

As a final example and an indication of the manner in which feedback signals can be used to improve the stability characteristics of a system, we shall find the root locus for the system of the previous example when $H(s)$ is changed from unity to $2s+1$ and determine the limits on α for stability. The alteration of $H(s)$ is equivalent to adding a feedback signal of $-2\dot{c}(t)$ to the unity-feedback system of Figure 10.5*a*, resulting in Figure 10.6*a*. Writing the transfer function $G(s)H(s)$ directly from the block diagram and then rearranging so as to identify K and $F(s)$, we have

$$G(s)H(s) = \underbrace{10\alpha}_{K}\underbrace{\left[\frac{s+0.5}{(s-1)(s^2+2s+5)}\right]}_{F(s)}$$

Although the poles of $F(s)$ have not been affected, there is now a zero at $z_1 = -0.5$ so $m = 1$, thereby reducing the number of large-gain asymptotes from three to two. Applying rule *6*, they make angles of $\pm 90°$ with the real axis and pass through the point

$$\sigma_0 = \frac{[1+(-1+j2)+(-1-j2)]-[-0.5]}{3-1} = -\frac{1}{4}$$

The real-axis portion of the locus lies between the pole at $s = 1$ and the zero at $s = -0.5$. Finally, the angles of departure from the complex poles can be computed. For example, a branch leaves p_2 with an angle of

$$\phi_2 = \underbrace{\arg\,[p_2 - z_1]}_{114°} - \underbrace{\arg\,[p_2 - p_1]}_{135°} - \underbrace{\arg\,[p_2 - p_3]}_{90°} + 180° = 69°$$

Having the above information at our disposal, we can sketch the salient features of the root locus, resulting in Figure 10.6*b*, or at least a close

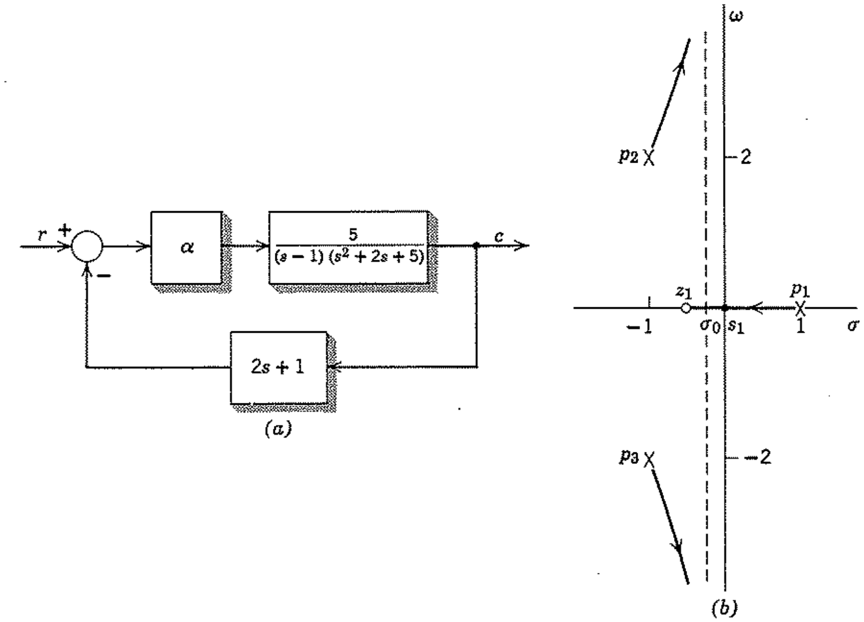

Figure 10.6. A third-order system with one zero. (*a*) Block diagram. (*b*) Root locus.

approximation thereof. Comparing the locus with Figure 10.5*b*, we see that the complex branches emanating from p_2 and p_3 no longer enter the RHP as α is increased. Thus, the addition of the zero – in physical terms, the addition of the rate-feedback signal – has had the effect of keeping the complex branches of Figure 10.5*b* in the LHP. Hence, the modified system will be stable for all values of $\alpha > \alpha_1$ which, upon applying the magnitude criterion with $s = 0$ and using the relationship $K = 10\alpha$, turns out to be $\alpha_1 = 1.00$.

10.2 DERIVATION OF BASIC ROOT-LOCUS RULES

In this section the basic rules introduced in the previous section for constructing plots of the 180° root locus will be derived.

Number of branches (Rule *1*)

From the version of the characteristic equation given in Eq. (9), Sect. 10.1, as the *sum* of polynomials of degree n and m, it is apparent that its degree is n when the condition $m \leq n$ is satisfied. Hence there will be